An Integer Programming Formulation of the Web Server Location Problem

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Abstract

We consider the problem of optimally placing web servers in the Internet so that the total cost is minimized subject to satisfying QoS constraints. The problem is stated and solved using a heuristic in our paper "Performance Analysis and Optimization of Web Proxy Servers and Mirror Sites", which is submitted for publication to *European Journal of Operational Research*. In this article, a simple 2×3 grid is considered and an Integer Programming formulation is provided.

Consider the simple 2×3 case with $A = 2$ and $B = 3$. The mathematical program is written in terms of the decision variables X_{ij} and other dummy variables $Y_{ij,kl}$ and $V_{ij,kl}$, where $Y_{ij,kl}$ is the demand in node (i,j) satisfied by server in (k,l), and $V_{ij,kl}$ is 1 if $Y_{ij,kl} > 0$ and 0 otherwise.

$$\text{Minimize} \quad \sum_{i=1}^{A} \sum_{j=1}^{B} C_{ij} X_{ij}$$

Subject to:

$$
\begin{aligned}
Y_{11,11} &= N_{11}X_{11} \\
Y_{11,12} &= N_{11}(1-X_{11})(1-X_{21})X_{12} + \lceil N_{11}/2 \rceil (1-X_{11})X_{12}X_{21} \\
Y_{11,13} &= N_{11}(1-X_{11})(1-X_{12})(1-X_{21})(1-X_{22})X_{13} + \lceil N_{11}/2 \rceil (1-X_{11})(1-X_{12})(1-X_{21})X_{22}X_{13} \\
Y_{11,21} &= N_{11}(1-X_{11})(1-X_{12})X_{21} + \lceil N_{11}/2 \rceil (1-X_{11})X_{21}X_{12} \\
Y_{11,22} &= N_{11}(1-X_{11})(1-X_{12})(1-X_{21})(1-X_{13})X_{22} + \lceil N_{11}/2 \rceil (1-X_{11})(1-X_{12})(1-X_{21})X_{13}X_{22} \\
Y_{11,23} &= N_{11}(1-X_{11})(1-X_{12})(1-X_{21})(1-X_{13})(1-X_{22})X_{23}
\end{aligned}
$$

$$
\begin{aligned}
Y_{12,11} &= N_{12}(1-X_{12})(1-X_{13})(1-X_{22})X_{11} + \lceil N_{12}/2 \rceil (1-X_{12})X_{11}(1-X_{13})X_{22} \\
&\quad + \lceil N_{12}/2 \rceil (1-X_{12})X_{11}(1-X_{22})X_{13} + \lceil N_{12}/3 \rceil (1-X_{12})X_{11}X_{22}X_{13} \\
Y_{12,12} &= N_{12}X_{12} \\
Y_{12,13} &= N_{12}(1-X_{12})(1-X_{11})(1-X_{22})X_{13} + \lceil N_{12}/2 \rceil (1-X_{12})X_{13}(1-X_{11})X_{22} \\
&\quad + \lceil N_{12}/2 \rceil (1-X_{12})X_{13}(1-X_{22})X_{11} + \lceil N_{12}/3 \rceil (1-X_{12})X_{11}X_{22}X_{13} \\
Y_{12,21} &= N_{12}(1-X_{11})(1-X_{12})(1-X_{13})(1-X_{22})(1-X_{23})X_{21} \\
&\quad + \lceil N_{12}/2 \rceil (1-X_{11})(1-X_{12})(1-X_{13})(1-X_{22})X_{21}X_{23} \\
Y_{12,22} &= N_{12}(1-X_{12})(1-X_{11})(1-X_{13})X_{22} + \lceil N_{12}/2 \rceil (1-X_{12})X_{22}(1-X_{11})X_{13} \\
&\quad + \lceil N_{12}/2 \rceil (1-X_{12})X_{22}(1-X_{13})X_{11} + \lceil N_{12}/3 \rceil (1-X_{12})X_{11}X_{22}X_{13} \\
Y_{12,23} &= N_{12}(1-X_{11})(1-X_{12})(1-X_{13})(1-X_{22})(1-X_{21})X_{23} \\
&\quad + \lceil N_{12}/2 \rceil (1-X_{11})(1-X_{12})(1-X_{13})(1-X_{22})X_{21}X_{23}
\end{aligned}
$$

$$
\begin{aligned}
Y_{13,11} &= N_{13}(1-X_{13})(1-X_{12})(1-X_{23})(1-X_{22})X_{11} + \lceil N_{13}/2 \rceil (1-X_{13})(1-X_{12})(1-X_{23})X_{22}X_{11} \\
Y_{13,12} &= N_{13}(1-X_{13})(1-X_{23})X_{12} + \lceil N_{13}/2 \rceil (1-X_{13})X_{12}X_{23} \\
Y_{13,13} &= N_{13}X_{13} \\
Y_{13,21} &= N_{13}(1-X_{13})(1-X_{12})(1-X_{23})(1-X_{11})(1-X_{22})X_{21} \\
Y_{13,22} &= N_{13}(1-X_{13})(1-X_{12})(1-X_{23})(1-X_{11})X_{22} + \lceil N_{13}/2 \rceil (1-X_{13})(1-X_{12})(1-X_{23})X_{11}X_{22} \\
Y_{13,23} &= N_{13}(1-X_{13})(1-X_{12})X_{23} + \lceil N_{13}/2 \rceil (1-X_{13})X_{23}X_{12}
\end{aligned}
$$

$$
\begin{aligned}
Y_{21,11} &= N_{21}(1-X_{21})(1-X_{22})X_{11} + \lceil N_{21}/2 \rceil (1-X_{21})X_{11}X_{22} \\
Y_{21,12} &= N_{21}(1-X_{21})(1-X_{22})(1-X_{11})(1-X_{23})X_{12} + \lceil N_{21}/2 \rceil (1-X_{21})(1-X_{22})(1-X_{11})X_{23}X_{12} \\
Y_{21,13} &= N_{21}(1-X_{21})(1-X_{22})(1-X_{11})(1-X_{23})(1-X_{12})X_{13} \\
Y_{21,21} &= N_{21}X_{21} \\
Y_{21,22} &= N_{21}(1-X_{21})(1-X_{11})X_{22} + \lceil N_{21}/2 \rceil (1-X_{21})X_{22}X_{11} \\
Y_{21,23} &= N_{21}(1-X_{21})(1-X_{22})(1-X_{11})(1-X_{12})X_{23} + \lceil N_{21}/2 \rceil (1-X_{21})(1-X_{22})(1-X_{11})X_{12}X_{23}
\end{aligned}
$$

$$
\begin{aligned}
Y_{22,11} &= N_{22}(1-X_{21})(1-X_{22})(1-X_{23})(1-X_{12})(1-X_{13})X_{11} \\
&\quad +\lceil N_{22}/2\rceil(1-X_{21})(1-X_{22})(1-X_{23})(1-X_{12})X_{11}X_{13} \\
Y_{22,12} &= N_{22}(1-X_{22})(1-X_{21})(1-X_{23})X_{12}+\lceil N_{22}/2\rceil(1-X_{22})X_{12}(1-X_{21})X_{23} \\
&\quad +\lceil N_{22}/2\rceil(1-X_{22})X_{12}(1-X_{23})X_{21}+\lceil N_{22}/3\rceil(1-X_{22})X_{21}X_{12}X_{23} \\
Y_{22,13} &= N_{22}(1-X_{21})(1-X_{22})(1-X_{23})(1-X_{12})(1-X_{11})X_{13} \\
&\quad +\lceil N_{22}/2\rceil(1-X_{21})(1-X_{22})(1-X_{23})(1-X_{12})X_{11}X_{13} \\
Y_{22,21} &= N_{22}(1-X_{22})(1-X_{23})(1-X_{12})X_{21}+\lceil N_{22}/2\rceil(1-X_{22})X_{21}(1-X_{23})X_{12} \\
&\quad +\lceil N_{22}/2\rceil(1-X_{22})X_{21}(1-X_{12})X_{23}+\lceil N_{22}/3\rceil(1-X_{22})X_{21}X_{12}X_{23} \\
Y_{22,22} &= N_{22}X_{22} \\
Y_{22,23} &= N_{22}(1-X_{22})(1-X_{21})(1-X_{12})X_{23}+\lceil N_{22}/2\rceil(1-X_{22})X_{23}(1-X_{21})X_{12} \\
&\quad +\lceil N_{22}/2\rceil(1-X_{22})X_{23}(1-X_{12})X_{21}+\lceil N_{22}/3\rceil(1-X_{22})X_{21}X_{12}X_{23}
\end{aligned}
$$

$$
\begin{aligned}
Y_{23,11} &= N_{23}(1-X_{23})(1-X_{22})(1-X_{13})(1-X_{21})(1-X_{12})X_{11} \\
Y_{23,12} &= N_{23}(1-X_{23})(1-X_{22})(1-X_{13})(1-X_{21})X_{12}+\lceil N_{23}/2\rceil(1-X_{23})(1-X_{22})(1-X_{13})X_{21}X_{12} \\
Y_{23,13} &= N_{23}(1-X_{23})(1-X_{22})X_{13}+\lceil N_{23}/2\rceil(1-X_{23})X_{13}X_{22} \\
Y_{23,21} &= N_{23}(1-X_{23})(1-X_{22})(1-X_{13})(1-X_{12})X_{21}+\lceil N_{23}/2\rceil(1-X_{23})(1-X_{22})(1-X_{13})X_{12}X_{21} \\
Y_{23,22} &= N_{23}(1-X_{23})(1-X_{13})X_{22}+\lceil N_{23}/2\rceil(1-X_{23})X_{22}X_{13} \\
Y_{23,23} &= N_{23}X_{23}
\end{aligned}
$$

$Y_{11,11}+Y_{12,11}+Y_{13,11}+Y_{21,11}+Y_{22,11}+Y_{23,11} \le \psi_{11}(4)+[\psi_{11}(3)-\psi_{11}(4)](1-V_{23,11})+[\psi_{11}(2)-\psi_{11}(3)](1-V_{23,11})(1-V_{22,11})(1-V_{13,11}) +[\psi_{11}(1)-\psi_{11}(2)](1-V_{23,11})(1-V_{22,11})(1-V_{13,11})(1-V_{12,11})(1-V_{21,11})$

$Y_{11,12}+Y_{12,12}+Y_{13,12}+Y_{21,12}+Y_{22,12}+Y_{23,12} \le \psi_{12}(3)+[\psi_{12}(2)-\psi_{12}(3)](1-V_{23,12})(1-V_{21,12}) +[\psi_{12}(1)-\psi_{12}(2)](1-V_{23,12})(1-V_{22,12})(1-V_{21,12})(1-V_{13,12})(1-V_{11,12})$

$Y_{11,13}+Y_{12,13}+Y_{13,13}+Y_{21,13}+Y_{22,13}+Y_{23,13} \le \psi_{13}(4)+[\psi_{13}(3)-\psi_{13}(4)](1-V_{21,13})+[\psi_{13}(2)-\psi_{13}(3)](1-V_{21,13})(1-V_{22,13})(1-V_{11,13}) +[\psi_{13}(1)-\psi_{13}(2)](1-V_{21,13})(1-V_{22,13})(1-V_{11,13})(1-V_{12,13})(1-V_{23,13})$

$Y_{11,21}+Y_{12,21}+Y_{13,21}+Y_{21,21}+Y_{22,21}+Y_{23,21} \le \psi_{21}(4)+[\psi_{21}(3)-\psi_{21}(4)](1-V_{13,21})+[\psi_{21}(2)-\psi_{21}(3)](1-V_{13,21})(1-V_{12,21})(1-V_{23,21}) +[\psi_{21}(1)-\psi_{21}(2)](1-V_{13,21})(1-V_{12,21})(1-V_{23,21})(1-V_{22,21})(1-V_{11,21})$

$Y_{11,22}+Y_{12,22}+Y_{13,22}+Y_{21,22}+Y_{22,22}+Y_{23,22} \le \psi_{22}(3)+[\psi_{22}(2)-\psi_{22}(3)](1-V_{13,22})(1-V_{11,22}) +[\psi_{22}(1)-\psi_{22}(2)](1-V_{13,22})(1-V_{12,22})(1-V_{11,22})(1-V_{23,22})(1-V_{21,22})$

$Y_{11,23}+Y_{12,23}+Y_{13,23}+Y_{21,23}+Y_{22,23}+Y_{23,23} \le \psi_{23}(4)+[\psi_{23}(3)-\psi_{23}(4)](1-V_{11,23})+[\psi_{23}(2)-\psi_{23}(3)](1-V_{11,23})(1-V_{12,23})(1-V_{21,23}) +[\psi_{23}(1)-\psi_{23}(2)](1-V_{11,23})(1-V_{12,23})(1-V_{21,23})(1-V_{22,23})(1-V_{13,23})$

For all $i \in [1,A]$, $j \in [1,B]$, $k \in [1,A]$, $l \in [1,B]$, and a large positive number $\overline{M}$,

$$
\begin{aligned}
Y_{ij,kl} &\le \overline{M}V_{ij,kl} \\
1-\overline{M}(1-V_{ij,kl}) &\le Y_{ij,kl}
\end{aligned}
$$

$$
X_{11}+X_{12}+X_{13}+X_{21}+X_{22}+X_{23} \ge 1
$$

For all $i \in [1,A]$, $j \in [1,B]$, $k \in [1,A]$, $l \in [1,B]$,

$$
X_{ij} = 0 \text{ or } 1
$$

$$Y_{ij,kl} = \text{non negative integer}$$
$$V_{ij,kl} = 0 \text{ or } 1$$

The above formulation for a small grid ($A = 2$ and $B = 3$) as an integer program with integer and binary variables and non-linear constraints results in a total of 78 (in general $AB + 2A^2B^2$) variables and 125 constraints (not including the 78 integer/binary constraints). The non-linearity needs to be removed using a standard linear transform for binary variables. For example, if $Z = UW$ where U and W are both 0 or 1, then the nonlinear condition can be broken down into three linear constraints $Z \leq U$, $Z \leq W$ and $Z \geq U + W - 1$. Similary, the above formulation can be converted to an integer program with linear constraints. However that would increase the number of variables to 158 and the number of constraints to 547 (not including the 158 integer/binary constraints).