Epidemic-based information dissemination in wireless mobile sensor networks

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***Abstract*—In this paper we consider wireless mobile sensor networks under extreme environments where nodes (i) have local knowledge, (ii) have limited computational power, (iii) make distributed decisions, and (iv) move rapidly over time. Information dissemination in these networks (or gossip) can be modeled via epidemic models that analyze behavior of the system mimicking the way diseases spread (or even gossip for that matter). However, the limitation on computational power and energy of nodes forces us to consider explicit stopping criteria that are seldom done in the literature. Furthermore, harsh environments considered in this paper prevent nodes from transmitting sensed information at specified time slots and hence might cause a large variation in inter-transmission time distribution. The objective of this paper is to characterize the dynamics of the information spread and obtain performance measures based on stochastic modeling. We start with modeling information flow using a Markov chain and obtain performance measures such as: time to transfer information and fraction of nodes receiving information. Then, we provide a method to obtain those performance measures when the assumption on inter-transmission time distribution is relaxed, e.g. time-varying transmission rates and non-exponential inter-transmission time distributions, which makes our model more realistic. We make a curious finding in that, for our proposed model, the average fraction of nodes receiving information is a parameter-free constant. We also show that our model is scalable and effective.**

***Index Terms*—epidemic models; wireless mobile networks; performance analysis; stochastic models; asymptotic analysis**

I. Introduction

In the near future, intelligent wireless mobile sensors will be extensively deployed in harsh environments such as military operations, under-sea explorations, hazardous environments, etc. The objectives for the nodes in such wireless mobile sensor networks are to move rapidly, probe, process and transmit information to other nodes. At the end of the "operation" only a subset of the sensor network nodes are recovered (the rest are either lost or severely damaged). Information is retrieved from what is stored in the subset of recovered nodes. Since the sensors have limited computational power, they have to balance between energy conservation and fault-tolerance while transmitting information. Epidemic information spreading models lend themselves well to such applications where each node selects at random a neighbor and transmits a quantum of information (referred to as *gossip*). There have been several research studies on epidemic information spreading models. Originally, studies have been conducted to analyze spreading mechanism of a disease. Most of previous studies in epidemics provide asymptotic analytical (mathematical) results when the number of nodes (or population) increases to infinity by utilizing functional version of Law of Large Number (see [1], [2]). According to Ball and Neal [3], there are three types of approaches to obtain limit processes in epidemics literature:

- Branching processes are used to model early stage of epidemics. Andersson [4] uses this approach for a discrete time model.
- The central limit theorem is used to obtain the number of infected people eventually. Anderson and Djehiche [5] shows weak convergence to a limit process for a continuous time model. We also apply the central limit theorem in Sections V and VI. We, however, use it to obtain a limit process for all time $t \geq 0$ not limited to one in steady state, i.e., when $t \to \infty$.
- Poisson approximations are extensively used to compute the number of susceptible people. Sellke [6], and Ball and Barbour [7] use this approach. Ball et al. [8], and Ball and Neal [3] apply this approach to obtain the number of susceptible people when they have two levels of contacts, i.e., global and local contacts.

All of the previous studies explained above assume that transition between states is synchronous or asynchronous with exponential transition distributions. For the general transition behavior, some mathematical expressions are derived to approximate average path behavior in [9]. However, it is hard to solve convolution integral equations with respect to general probability measures, and the expressions to obtain covariance matrix are not provided.

Due to the scalability and stability of epidemic models, they have been adopted as an effective method to disseminate information over large-scale communication networks. Zhang et al. [10] provides deterministic ODE models that are in fact the asymptotic results of Markovian models for epidemic routing problems. However, instead of analyzing the (asymptotic) phenomena, studies in communication networks are mostly concerned with developing fast and reliable algorithms to transfer information to as many nodes as possible under fixed network topology. Eugster el al. [11] summarizes epidemic models in communication networks and emphasizes the easiness of deployment, robustness and stability. Boyd et al. [12] and Mosk-Aoyama et al. [13] apply gossip algorithms to solve distributed computing problems (separable functions) considering both synchronous and asynchronous (with expo-

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nential inter-transmission time distributions) models. Weber et al. [14] analyzes the performance of a particular gossip algorithm suitable for fixed networks using copulas. However, besides using fixed networks, the selection of right copulas is still an open question. Haas and Small [15] and Hass et. al [16] use epidemic models for a routing problem in ad-hoc networks. Some rules of thumb to reduce the number of transmissions are suggested from a number of experiments. Sistla et al. [17] considers layered gossip networks using simulation. Deb et al. [18] looks at the gossip algorithm from an information theoretic standpoint. When multiple messages are transmitted in the network, the number of transmission to deliver messages is reduced by proposed network coding method.

In designing and constructing an epidemic information spreading model for wireless mobile sensor networks, we found that there are three main problems that are not sufficiently addressed by previous studies. First, stopping criteria under which each node stops transmitting data are hardly taken into consideration in the literature. This could become an important issue when the nodes take multiple consecutive missions with limited energy sources. In fact, the susceptible-infective-removed (SIR) model in the epidemics literature has a similar concept of stopping criteria, that is, people who were infected but cured are never infected again. The stopping rule in the SIR model can be a meaningful way to stop information dissemination. However, this stopping rule seems to be effective from a receiver's perspective as the node that received the information will not receive the information any longer. Therefore, some modification is required to apply the stopping rule in the SIR model to our settings, and our stopping criterion that will be explained in Section II-A is a somewhat modified version of that. Second, only a few previous studies have considered the mobility of sensors. Although Hass et. al [16] mentioned the mobility of nodes, a random graph structure that they consider to reflect mobility is appropriate for slowly moving sensors since network topology does not change once transmission of information begins. Third, transmission behavior might be affected by external environment. Most of previous studies assumed that each transmission occurs at fixed time slots (synchronous time) or inter-transmission time follows exponential distribution (asynchronous time). Although exponential inter-transmission time is justified by Groenevelt et al. [19], it may not be appropriate for harsh environment, where it could be time-varying or have a large variation.

Keeping these in mind, our paper has discriminating features from the previous studies as follows:

- We provide analytical techniques to compute the performance of information dissemination in wireless mobile sensor networks. We incorporate a stopping criterion and evaluate its effect on performance measures.
- Mobility is integrated into our model. We consider a simple mobility model similar as the independent and identically distributed (i.i.d.) model used to analyze the capacity-delay tradeoffs in mobile ad-hoc networks ([20], [21], [22], [23]). For more refined models, refer to Groenevelt et al. [19].
- Most importantly, we obtain performance measures when the assumption on inter-transmission time distribution is relaxed, e.g. time-varying transmission rates or general inter-transmission time distributions, which have been hardly addressed in previous studies but are essential to analyze the system considered in this paper.

Elaborating on the last bullet above, for the time-varying transmission rates, we employ fluid and diffusion approximations studied in [24], [25], [26], and [27]. The main idea of fluid and diffusion approximations in the literature listed above is to obtain limit processes involving Brownian motion by suitably accelerating rate functions (e.g. transmission rate in our paper). While diffusion approximation requires strict conditions that are not satisfied even in popular queueing systems, we will show that our model satisfies those conditions in Section V-B. For general inter-transmission time distributions, we adopt techniques that approximate any distribution function having positive support with phase-type distributions since they are proven to be dense in all distributions with positive support ([28]) and have *nice* properties. One common method to find a phase-type distribution is matching moments due to its easiness and convenience ([29], [30], [31], [32]). It, however, has limitation since it may lose properties of target distribution. Therefore, in this paper, we use the methodology in [33] that approximates the distribution function itself and is applicable even if no moment exists.

The objectives of this paper are: (i) to introduce an information dissemination model with explicit stopping criteria for wireless mobile sensors under harsh environments; (ii) to analytically model the system dynamics to obtain expressions for performance measures; (iii) to investigate if any insights can be obtained for the performance measures, and if any techniques can be developed to predict the performance for a given number of nodes in the network; (iv) to study the flexibility and scalability of both proposed model and its performance analysis, specifically how they work as the number of nodes in the network increases and transmission behavior changes.

In that light, the remainder of this paper is organized as follows. In Section II we explain our model suitable for applications described in the first paragraph above. Then in Section III we start with a synchronous time model (i.e. the transmission time of each node is discrete and occurs at the same time for all nodes) using a discrete time Markov chain. By suitably modifying this Markov chain, we analyze its performance to obtain measures such as: time to transfer information and fraction of nodes receiving information. We describe numerical results and findings. After that, we consider three asynchronous time models (i.e. inter-transmission time of each node is independent and randomly distributed). In Section IV, we construct a model using a continuous time Markov chain, i.e. inter-transmission time is exponentially distributed. Then we provide two types of relaxations in Sections V and VI. In Section V, we show the results for the time-varying transmission rate which can be useful to model systems such as military operations (e.g. day-time hiding and night-time operating). We consider, in Section VI, general inter-transmission time distribution other than exponential distribution and show how performance measures can be obtained. Finally we present our concluding remarks and directions for future work in Section VII.

II. Model Description

In this section, we explain our model and define the performance measures for it. In Section II-A, we raise a question for the effects of transmission behavior in epidemic-based information dissemination and explain the various models we consider. Then, to evaluate the performance of each model, we define two performance measures in Section II-B.

A. Epidemic-based Information Dissemination Models

Given a wireless mobile sensor network with N nodes, consider an information dissemination model for nodes to dissipate sensed information to as many nodes as possible. We specifically concentrate on a scenario where the sensor nodes move around, they sense and process information that they periodically transmit to other nodes. The inter-transmission times are far apart that the nodes would have significantly moved between two transmissions. Between successive transmissions, the nodes sense, process and store information.

One question that comes to mind is: should the transmissions be made at given time slots or should they be made at random? If random, what is the effect of inter-transmission time distributions? This issue has been around in multiple access communications for quite a while. Although the scenario considered in this paper is different, it is worthwhile to investigate the difference in performance for various transmission behavior. Let $F(\cdot)$ denote the cumulative distribution function of inter-transmission time of each node. For information transmission, we consider various cases of this function and call it $F(u)$. In Section III, we construct a synchronous time model when $F(u)$ is a step function, i.e. constant inter-transmission time. An asynchronous time model ($F(u) = 1 - e^{-u}$) described in [34] and used in [35] as well as [36] is built in Section IV. Furthermore, we consider more general asynchronous time models and investigate whether and how the performance of our model is affected by various inter-transmission time distributions ($F(u) = 1 - e^{-r(t)u}$ or $F(u) = G(u)$ for some heavy-tailed cdf $G(u)$) in Sections V and VI.

In the synchronous time model, all nodes at a prespecified transmission time, synchronously transmit a set of information to one of their neighbors. Each node selects one neighbor and transmits only the information that the neighbor does not possess. To describe the algorithm as well as for analysis, we will consider only a single piece of information that we call *gossip* to study how fast and wide it can spread. In the asynchronous time model, inter-transmission time of each node is random with mean time $\frac{1}{\lambda}$, and the other assumptions are the same as those of the synchronous time model.

Note that the nodes (i) have local knowledge, (ii) have limited computational power, (iii) make distributed decisions, and (iv) move rapidly over time. With those in mind, we arrive at the following information dissemination model. Consider a gossip that was originated at a certain node. During the next transmission time for the node, the node picks its closest neighbor to transmit the gossip. Since the nodes are moving rapidly, we assume that the probability that a certain node is selected (i.e. closest neighbor) is $\frac{1}{N-1}$, even though all nodes may not be candidate neighbors. At the next transmission time for each of these two nodes that know the gossip, the transmitting node selects a neighboring node at random with probability $\frac{1}{N-1}$. Although not exactly the same but a similar mobility model is provided in [23] known as i.i.d mobility model. They assume a cell-partitioned network and each node chooses a cell randomly to move at next time slot. This implies *infinite mobility* that is reasonable for a network in which nodes are moving quickly relative to inter-transmission times. Moreover, from Theorem III-4 in [20], for a large population, our selection mechanism could be justified. In this manner, during every transmission time of a node, every node that has the gossip (and has not stopped spreading it) selects one of the $N-1$ other nodes to check if it has the gossip. If the selected node already has the gossip, then the transmitting node not only does not transmit the gossip but also stops spreading it; else it continues spreading the gossip. If the size of gossip is small, it would be reasonable to skip the checking procedure and determine whether to continue spreading it based on the acknowledgment. As a result of this explicit stopping criterion, each node stops spreading the gossip and conserves energy. *Note:* stopping the gossip spreading implies stopping the checking procedure as well. Therefore, a node does not spend network resources and energy for the gossip any longer once it decides to stop spreading.

In the synchronous time model, if two or more nodes attempt to transmit the gossip to a node that does not have the gossip, then only one of the nodes transmits the gossip but all the nodes involved continue to spread the gossip. In the asynchronous time models, we assume that the transmission between any two particular nodes lasts for such a small time that the probability that another node begins transmission to one of these two particular nodes during their information exchange is zero.

B. Performance Measures

It is worthwhile to notice the following characteristics with respect to the proposed gossip algorithm: (i) there is an explicit stopping criterion for each node to stop spreading the gossip, i.e. when the node attempts to transmit to another node that already has the gossip; (ii) at each transmission phase, there are three types of nodes, those that are actively spreading the gossip, those that stopped spreading the gossip and those that do not have the gossip; (iii) the algorithm ends when there are no actively spreading nodes with gossip, and it is not necessary that all nodes get the gossip eventually; (iv) when the algorithm ends, the number of transmissions that occurred is one less than the number of nodes that have the gossip (thus computing power is used prudently). For such a system, we define two different metrics that are, in fact, similar to metrics defined in previous research studies. The first performance measure is τ which is the average number of time slots (synchronous time model) or the average time (asynchronous time model) for completion: i.e., elapsed time to stop spreading the gossip. It is actually used to measure how fast our algorithm is. The second measure of interest is fraction of nodes that end up getting the gossip when the algorithm completes. We call it *reach*. Let μ and σ be the average and standard deviation of reach, i.e. the mean and standard deviation of the fraction of nodes that receive the gossip. These μ and σ are used to assess how reliable our models are. Therefore, from the metrics defined above, we are going to investigate how fast and reliable our models are and how they affected by transmission behavior. To distinguish performance measures of each model, we will use a subscript to specify each model, e.g. τ_d, μ_d and σ_d are for the synchronous time model (d stands for discrete time).

III. Constant inter-transmission time ($F(u)$ is a step function)

In this model, time between when active nodes begin transmitting gossip is taken as 1 time unit. Therefore nodes begin transmission at times 1, 2, 3, ..., until the algorithm ends.

A. Discrete time Markov model

Let X_n be the number of actively spreading nodes at the n^{th} transmission phase, Y_n be the number of nodes that have not heard the gossip up to the n^{th} transmission phase, and Z_n be the number of nodes that have heard the gossip until the n^{th} transmission but have stopped spreading it. Note that since $Z_n = N - X_n - Y_n$, we really need only X_n and Y_n to describe the state of the system, and Z_n is defined purely for notational convenience. Also, $X_0 = 1$, $Y_0 = N-1$ and $Z_0 = 0$ is the initial state when one node has the gossip and the remaining $N-1$ do not know it. Further, in order to predict the state (X_{n+1}, Y_{n+1}) all we need to know is (X_n, Y_n). Therefore the process $\{(X_n, Y_n), n \geq 0\}$ is a discrete time Markov chain (DTMC).

For this DTMC model, the next step is to obtain the transition probabilities. Let $p_{i,j}(i-l+m, j-m)$ be the transition probability from state $(X_n = i, Y_n = j)$ to state $(X_{n+1} = i-l+m, Y_{n+1} = j-m)$. That happens when l of the i nodes that are active in spreading the gossip end up attempting to spread to nodes that already have the gossip. Also, the remaining $i-l$ nodes spread the gossip to a set of m nodes out of the j nodes that do not have the gossip. There are some constraints such as $0 \leq m \leq \min(j, i-l)$ and $0 \leq l \leq i$. In order to obtain an algebraic expression for $p_{i,j}(i-l+m, j-m)$, consider the following matching problem: there are W eligible women and M eligible men in a society, and each of the W women selects a man at random (assuming all men are equally desirable) and writes a letter.

Lemma 1: The number of combinations of m of the M men receiving letters (that means $M-m$ do not receive letters) is $\Omega(M, W, m)$ and is given by

$$\Omega(M, W, m) = \binom{M}{m}\left[\sum_{x=1}^{m}(-1)^{m-x}\binom{m}{x}x^W\right].$$

Proof: The first term $\binom{M}{m}$ is just the number of different ways m men can be selected from M. Next, given a specific set of m men, $\left[\sum_{x=1}^{m}(-1)^{m-x}\binom{m}{x}x^W\right]$ is the number of different ways W women can send letters to this specific set of m men. The above expression comes out of inclusion-exclusion principle often adopted in set theory. Consider finite sets $A_1, A_2, \ldots, A_m$. For $i \in \{1, 2, \ldots, m\}$, each A_i represents a set of events where W women send letters to i out of m men. Then, $\left[\sum_{x=1}^{m}(-1)^{m-x}\binom{m}{x}x^W\right]$ is the cardinality of $\bigcup_{i=1}^{m} A_i$. Therefore, essentially it is derived from

$$\begin{aligned} n\{A_1 \cup A_2 \cup \ldots \cup A_m\} &= \sum_{i=1}^{m} n\{A_i\} - \sum_{i,j:i>j} n\{A_i \cap A_j\} \\ &+ \sum_{i,j,k:i>j>k} n\{A_i \cap A_j \cap A_k\} - \ldots \\ &+ (-1)^{m-1} n\{A_1 \cap A_2 \cap \ldots \cap A_m\}, \end{aligned}$$

where $n\{A\}$ is the cardinality of set A. ■

Such combinatorial problems have received attention in the literature, especially the stable marriage problem (see Gusfield and Irving [37]). Now, using the above Lemma 1, we can obtain an algebraic expression for $p_{i,j}(i-l+m, j-m)$. Let $Z_n = k$ where k is the number of nodes that have stopped spreading the gossip. Clearly, $k = N-i-j$. Also $X_{n+1} = i-l+m$ and $Y_{n+1} = j-m$ imply that $Z_{n+1} = k+l$. The following theorem describes the transition probability $p_{i,j}(i-l+m, j-m)$.

Theorem 1: For $m = 0$, (and $l = i$ hence)

$$p_{i,j}(i-l+m, j-m) = \left(\frac{i+k-1}{N-1}\right)^l$$

and for $1 \leq m \leq \min(j, i-l)$ and $0 \leq l \leq i$,

$$p_{i,j}(i-l+m, j-m) =$$

$$\binom{i}{l}\left(\frac{i+k-1}{N-1}\right)^l\left(\frac{j}{N-1}\right)^{i-l}\frac{\Omega(j, i-l, m)}{j^{i-l}}.$$

Proof: The event that l nodes stop spreading the gossip is when l of the i nodes that are actively spreading the gossip end up spreading to nodes that already know the gossip. In other words, out of the i nodes, l of them choose to tell nodes who already know the gossip, and $i-l$ of them choose to tell nodes that do not have the gossip. That happens with probability

$$\binom{i}{l}\left(\frac{i+k-1}{N-1}\right)^l\left(\frac{j}{N-1}\right)^{i-l}.$$

Now, of the j nodes that have not received the gossip, m of them get it from the $i-l$ nodes mentioned above. Clearly some (or all) of the m nodes may have been contacted by more than one of the $i-l$ nodes. The total number of different ways j nodes can get the gossip from $i-l$ nodes is j^{i-l} ways. Of these j^{i-l} ways, based on Lemma 1, exactly $\Omega(j, i-l, m)$ would result in m different new nodes hearing the gossip, where

$$\Omega(j, i-l, m) = \binom{j}{m}\left[\sum_{x=1}^{m}(-1)^{m-x}\binom{m}{x}x^{i-l}\right].$$

Hence the expression for the transition probability. ■

Care must be taken while constructing the DTMC to ensure that impossible states are removed from the DTMC. For example $(1, N-1)$ is the only state for which $X_n = 1$. Also there cannot be states such as $(0, N)$ or $(0, N-1)$. In essence, all states that are unreachable from $(1, N-1)$ in one or more steps should be removed. Let P be the transition probability matrix of the DTMC such that it is a matrix of $p_{i,j}(i-l+m, j-m)$ values. Now that the system is modeled as a DTMC with transition probabilities from Theorem 1, the next step is to obtain performance measures such as time for gossip spreading to end and fraction of nodes receiving the gossip.

B. Performance Analysis

The DTMC modeled in Section III-A is reducible and transient, since states such as $(0, j)$ are absorbing (i.e. $p_{0,j}(0, j) = 1$) and state $(1, N-1)$ cannot be reached from any other state. Although it is possible to analyze reducible DTMCs, for the purposes of this paper, it is more convenient to transform the DTMC into an irreducible and positive recurrent one (such DTMCs are sometimes called ergodic, for definition and properties refer to Kulkarni [38]).

1) Modified DTMC: Consider a modification to the transition probability matrix P such that for all $j \leq N-2$, $p_{0,j}(0, j) = 0$ and $p_{0,j}(1, N-1) = 1$. Let $\hat{P}$ be the new transition probability matrix, but with the same states as the original DTMC. The modification implies that as soon as a gossip spreading ends, a new gossip begins. Therefore every time state $(1, N-1)$ is reached, it is like starting a new replication in a simulation. This modified $\hat{P}$ matrix is such that the DTMC is irreducible and aperiodic. Note that every time the system reaches $(1, N-1)$, it denotes the gossip spreading ended in the previous transition. Also if $(1, N-1)$ was reached from $(0, j)$ then $N-j$ nodes ended up receiving the gossip. In that light the two performance measures we are interested are: how long does it take to revisit $(1, N-1)$ (i.e. how many transmission slots does the gossip algorithm last which is a function of how often state $(1, N-1)$ occurs); and how many nodes end up receiving the gossip.

For that, we compute the steady-state distribution of the modified DTMC. Let $\pi_{i,j}$ be the steady-state probabilities of the DTMC with transition probability matrix $\hat{P}$. Therefore,

$$\pi_{i,j} = \lim_{n \to \infty} P\{X_n = i, Y_n = j\}.$$

Clearly the row vector π of $\pi_{i,j}$ values ($\pi = [\pi_{i,j}]$) can be obtained as the left eigen vector of $\hat{P}$ corresponding to eigenvalue of 1 and normalized so that $\sum_{i,j} \pi_{i,j} = 1$. In other words, π is the solution to

$$\pi = \pi \hat{P} \text{ and } \pi e = 1$$

where e is a column vector of 1's. Note that $\pi_{i,j}$ is not only the probability that the DTMC is in state (i, j) in steady state, but it is also the fraction of time the DTMC spends in state (i, j) in the long run. Now, using the $\pi_{i,j}$ values and their interpretation, we develop the performance measures of interest in the next section.

2) Performance Measures: For all states (i, j) in the DTMC, $\pi_{i,j}$ can be obtained numerically. Using that, we derive τ, μ, and σ for the DTMC model in terms of $\pi_{i,j}$ (we use τ_d, μ_d, and σ_d respectively to indicate the DTMC model). Then, from the following theorem, we obtain τ_d.

Theorem 2: $\tau_d = \dfrac{1}{\pi_{1,N-1}} - 1.$

Proof: For the modified DTMC, since state $(1, N-1)$ is reached every time the gossip spreading is completed, $1/\pi_{1,N-1}$ denotes the average amount of time to leave state $(1, N-1)$ and return back for the first time. This is one extra transmission phase over when the gossip ended. Hence the average time for gossip to end in terms of number of transmission slots is given in Theorem 2. ■

We can also obtain μ_d and σ_d as follows:

$$\mu_d = \frac{\sum_{r=2}^{N} r \rho_{N-r}}{N}$$

$$\sigma_d^2 = \frac{\sum_{r=2}^{N} r^2 \rho_{N-r} - \mu_d^2}{N^2}$$

where ρ_j is the probability that j nodes do not receive the gossip when the algorithm ends. In order to compute ρ_j, we state the following theorem.

Theorem 3: $\rho_j = \dfrac{\pi_{0,j}}{\sum_{i=0}^{N-2} \pi_{0,i}}.$

Proof: Straightforward conditional probability that the algorithm ends in state $(0, j)$ given that the DTMC is in one of the algorithm-ending nodes, i.e. of the form $(0, i)$. ■

Having derived the performance measures τ_d, μ_d and σ_d, the next step is to obtain them numerically.

C. Results

Notice that N is the only parameter in the DTMC modeled in Section III-A. Therefore for various values of N, we obtain the performance measures τ_d, μ_d and σ_d numerically. For each N, we obtain P first, then convert to $\hat{P}$ and finally the steady-state probabilities $\pi_{i,j}$. Using the steady-state probabilities, we obtain performance measures τ_d, μ_d and σ_d.

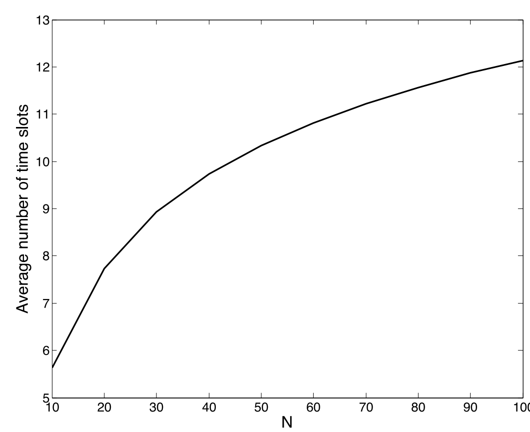


Fig. 1. τ_d versus N

Fig. 1 is an illustration of how τ_d, the average number of transmission phases for the gossip algorithm to complete, varies with N. From the figure, it is evident that as the number of nodes N increases, τ_d increases at a much slower rate and the rate of increase decreases with N. We can see similar results in Sections V and VI for the cases of a large number of nodes, and this implies that the proposed gossip algorithm is fairly scalable.

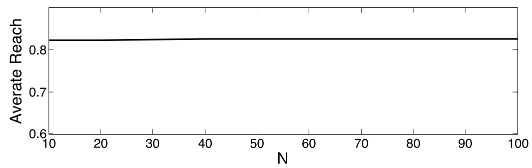


Fig. 2. μ_d versus N

The most remarkable finding of this paper is that in Fig. 2. Notice that the average fraction of nodes reached (μ_d) is almost independent of N. In fact, numerically it can be shown that $\mu_d \approx 0.82$ for $N \geq 5$. What this implies is that for any N, μ_d can be immediately predicted without even running the algorithm. In other words, on an average 82% of the nodes can be reached via the gossip algorithm presented in this paper. Notice that 0.82 is a parameter-free constant. Although we do not numerically verify it for $N > 100$ due to exponential growth of P matrix, we will justify it indirectly in Section V.

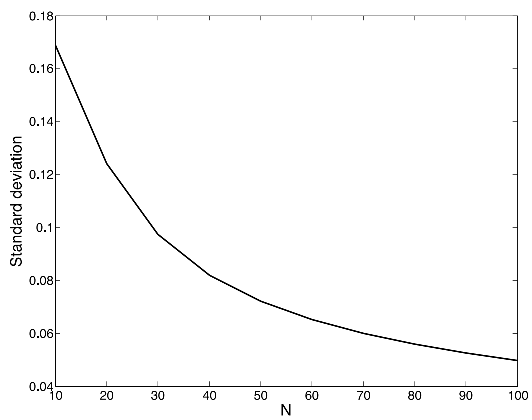


Fig. 3. σ_d versus N

In Fig. 3, we plot σ_d (the standard deviation of reach) as a function of N. Clearly that quantity, i.e. σ_d is a decreasing function of N. An interesting observation one can make is that with 99% confidence one can state that at least two-thirds of the nodes would receive the gossip, especially for large N. The claim is made using central limit theorem and strong law of large numbers. Therefore, in this section, for the gossip algorithm with the synchronous time model, we have examined performance measures such as algorithm spreading time and reach as well as studied their effects and made curious findings. Now we move to the asynchronous time model in the next section.

IV. Exponential inter-transmission time ($F(u) = 1 - e^{-u}$)

In the asynchronous time model, the time between when active nodes begin transmission of gossip is random. Therefore each node when it receives the gossip, begins transmission at times $T_1, T_1+T_2, T_1+T_2+T_3, \ldots$, until it sends the gossip to a node that already has it. Note that T_1, T_2, T_3, $\ldots$, are i.i.d. random variables with the same distribution for all nodes. In this section, we exhibit the simplest case when all T_i's are i.i.d. exponential random variables and extend it to more general cases in Section VI. Since the analysis procedure is similar to the DTMC case in Section III, we explain this model mainly focusing on the difference.

A. Continuous time Markov model

Let $X(t)$ denote the number of actively spreading nodes at time t; let $Y(t)$ denote the number of nodes that have not heard the gossip up to time t and let $Z(t)$ be the number of nodes that have heard the gossip until time t, but have stopped spreading it. Since $Z(t) = N - X(t) - Y(t)$, we need only $X(t)$ and $Y(t)$ to describe the state of the system. The initial state of the system is $X(0) = 1$, $Y(0) = N - 1$ and $Z(t) = 0$. Inter-transmission time of each node is independent and identically distributed according to an exponential distribution with mean time $\frac{1}{\lambda}$. Then, the process $\{(X(t), Y(t)), t \geq 0\}$ is a continuous time Markov chain (CTMC).

For this CTMC model, the infinitesimal generator matrix can easily be obtained. Let Q be the infinitesimal generator matrix of the system states, $q_{i,j}(i+1, j-1)$ be the element of Q which corresponds to the transition rate from state $(X(t) = i, Y(t) = j)$ to state $(X(t) = i+1, Y(t) = j-1)$, and $q_{i,j}(i-1, j)$ be the element of Q corresponding to the rate from state $(X(t) = i, Y(t) = j)$ to state $(X(t) = i-1, Y(t) = j)$. Note that $q_{i,j}(i+1, j-1)$ is the rate corresponding to when an actively transmitting node chooses a node that has not heard the gossip. In this case, the number of actively transmitting nodes increases by one and the number of nodes that have not heard the gossip decreases by one. However, $q_{i,j}(i-1, j)$ is the rate that an actively transmitting node chooses a node that already has known the gossip. In this case, the number of actively transmitting nodes decreases by one and the number of nodes that have not heard the gossip does not change. Note that these are the only two $q_{i,j}(\cdot, \cdot)$ that are positive. With that understanding, we state the following theorem that characterizes the infinitesimal generator matrix Q as follows: For $1 \leq i \leq N-1$ and $0 \leq j \leq N-i$,

$$
\begin{aligned}
q_{i,j}\big(i+1, (j-1)^+\big) &= \frac{j}{N-1} \cdot i \cdot \lambda \quad \text{and} \\
q_{i,j}(i-1, j) &= \frac{N-j-1}{N-1} \cdot i \cdot \lambda.
\end{aligned}
$$

Similar to the DTMC model in Section III, the states that cannot be reached from the state $(1, N-1)$ should be removed from the state space of the CTMC model as well. After adjusting the state space, we analyze the CTMC to obtain the performance measures defined earlier in the synchronous time model (Section III) with the infinitesimal generator matrix obtained above.

B. Performance Analysis

Consider the infinitesimal generator matrix Q of this system. Clearly, Q is reducible and transient (since states such as

$\{(0,j), 0 \leq j \leq N-2\}$ are absorbing and state $(1, N-1)$ cannot be reached from any other state). For the purpose of this paper, we transform the CTMC into an irreducible and positive recurrent one.

1) Modified CTMC: Similar to the DTMC model, as soon as a gossip spreading ends, we modify the CTMC so that a new gossip spreading begins. It implies the state transition rate from the states that cause a gossip spreading to end to the state $(1, N-1)$ is infinite. However, we use a sufficiently large value $\theta \in \{t : t < \infty, t \in \mathbb{R}^+\}$ instead of infinity since using infinity can causes the infinitesimal generator matrix to be reducible again. Let $\hat{Q}$ be the modified infinitesimal generator matrix that has same states as the original CTMC, and $p_{i,j}$ denote steady-state probabilities of the modified CTMC, i.e.,

$$p_{i,j} = \lim_{t \to \infty} P\{X(t) = i, Y(t) = j\}.$$

Then, the row vector p of $p_{i,j}$ values ($p = [p_{i,j}]$) is the solution to

$$p\hat{Q} = 0 \text{ and } pe = 1,$$

where e is a column vector of 1's. It is important to note that for an ergodic CTMC, p_{ij} can be interpreted as: (i) the probability that the CTMC is in state (i,j) in steady state; (ii) the fraction of time the CTMC spends in state (i,j) in the long run. Using these interpretations, we next develop some performance measures for the system in terms of p_{ij}.

2) Performance Measures: In this section, we obtain the performance measures (τ, μ, and σ) for the CTMC model. The subscript "c" is used here for CTMC (and is equivalent to the subscript "d" used for DTMC in Section III-A), i.e., we use τ_c, μ_c, and σ_c respectively.

The following theorem provides a formula to obtain τ_c.

Theorem 4: $\tau_c = \frac{1}{p_{1,N-1}} \cdot \frac{1}{\lambda} - \frac{1}{\theta}$.

We can obtain μ_c and σ_c in the exactly same way as the synchronous time model by replacing $\pi_{i,j}$ with $p_{i,j}$. Here the logic used is that p_{ij} is the probability that in the long run the system is in state (i,j). Given that the system is in one of the end states of the original algorithm, the probability it is in a specific end state $(0,j)$ is ρ_j. Thus:

$$\mu_c = \frac{\sum_{r=2}^{N} r\rho_{N-r}}{N} \text{ and}$$
$$\sigma_c^2 = \frac{\sum_{r=2}^{N} r^2\rho_{N-r} - \mu_c^2}{N^2}$$

where $\rho_j = \frac{p_{0,j}}{\sum_{i=0}^{N-2} p_{0,i}}$.

In the next section, we numerically obtain performance measures τ_c, μ_c, and σ_c.

C. Results

Unlike the synchronous time model, here in the asynchronous time model, λ as well as N are parameters. However, in order to compare with the synchronous time model, we fix the value of $1/\lambda$ as 1 time unit: therefore one transmission per unit time on average for each node actively spreading the gossip. For each N, we obtain $\hat{Q}$, $p_{i,j}$, and performance measures $(\tau_c, \mu_c$, and $\sigma_c)$ numerically.

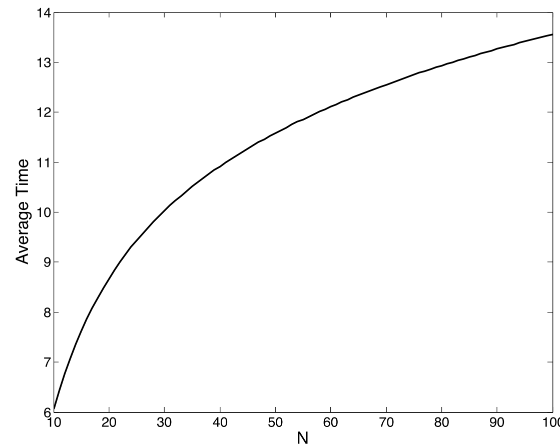


Fig. 4. τ_c versus N

Fig. 4 illustrates how τ_c changes according to N. The increasing rate of τ_c becomes slower as N increases. This behavior is very similar to the synchronous time model described in section III-A. Figs. 5 and 6 show very similar results to those corresponding to the synchronous time model. The average fraction of nodes reached (μ_c) is also independent of N for large N. It can be shown that $\mu_c \approx 0.79$ numerically,

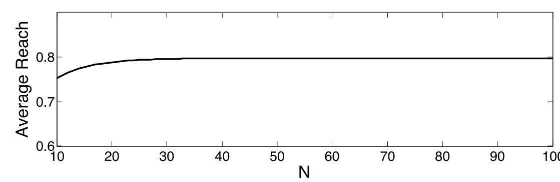


Fig. 5. μ_c versus N

and it implies that on an average 79% of the nodes can be reached via this asynchronous time model. Similar to the

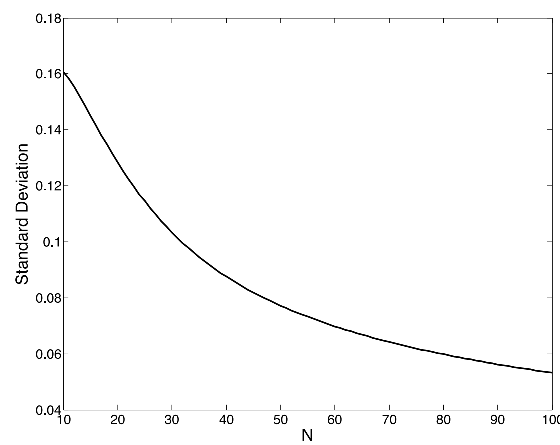


Fig. 6. σ_c versus N

synchronous time model, σ_c is a decreasing function of N and also at least two-thirds of nodes would eventually receive the gossip with 99% of confidence. Furthermore, the values of σ_c are extremely small. This implies that the random variable that indicates the fraction of nodes that receive the gossip is less varying in the asynchronous time model as well.

So far we have obtained analytical expressions and numerical solutions for the synchronous and asynchronous time models. We, however, notice two potential problems in our models. First, under harsh environments we consider, the synchronous time model may not be possible. In addition, even considering the asynchronous model, the exponential inter-transmission time distribution may not be appropriate to describe real system dynamics. Second, numerical methods for performance measures are not scalable although the models themselves look scalable. For the numerical solutions, we are able to obtain performance measures when $N = 100$ since the size of the state space increases by $O(N^2)$, which causes huge growth of the size of the transition matrix. In reality, however, the number of nodes could be nearly several millions ([39]). This implies that numerical methods we use until now would not be applicable for the cases having large population.

In order to resolve these limitations, we are going to utilize asymptotic methods called "strong approximations" proposed and developed by [24] and [26]. In the following sections, we will show how the asymptotic methods with some adjustment can be applied to our models and resolve problems.

V. Time-varying transmission rate ($F(u) = 1 - e^{r(t)u}$)

In this section, we investigate how to obtain performance measures when the transmission rate is time-varying. Before doing it, we first show that mean and standard deviation of reach (i.e. μ and σ) are independent of inter-transmission time distributions in Section V-A, which makes our analysis much simpler. Then, we provide how the remaining measure τ changes when the transmission rate is time-varying.

A. Invariance of μ and σ

If inter-transmission time distributions are i.i.d. exponential distributions, for small N, we can obtain performance measures numerically without difficulty. Even if N is large, we can also approximate them by utilizing asymptotic methods that will be explained in Section V-B. However, if inter-transmission times are not exponential random variables, in general, it is hard to obtain them. Recall that we have three performance measures, i.e. τ, μ, and σ. Fortunately, we found that two of them can be obtained using the results of exponential cases by the following theorem.

Theorem 5: Let $\mathbb{F}$ be a set of all continuous cdfs defined in $(0, \infty)$ with finite means. For a network with a fixed number of nodes, N, suppose $\mu = \mu_0$ and $\sigma = \sigma_0$, when the cdf of the inter-transmission times is $F_0 \in \mathbb{F}$. Then,

$$\mu = \mu_0 \text{ and } \sigma = \sigma_0 \quad \forall F \in \mathbb{F},$$

where F is a cdf of inter-transmission times.

Proof: We prove this by considering the embedded DTMC. If the cdf of inter-transmission times is continuous, at most one node can transmit the information at a time with probability 1. Let S_n be the time of the n^{th} transmission. Define embedded states (X^e_n, Y^e_n) denoting the state of the system just before the n^{th} transmission, i.e.

$$X^e_n = X(S^-_n) \text{ and } Y^e_n = Y(S^-_n).$$

Then, the probability transition matrix $P^e = [p^e_{(i,j)}(i', j')]$ is given as follows:
For $1 \le i \le N-1$ and $0 \le j \le N-i$,

$$\begin{aligned} p^e_{(i,j)}(i+1, (j-1)^+) &= \frac{j}{N-1} \quad \text{and} \\ p^e_{(i,j)}(i-1, j) &= \frac{N-j-1}{N-1}. \end{aligned}$$

Notice that P^e does not depend on the inter-transmission time distributions. Therefore, we prove the theorem. ■

From the above theorem, we prove that μ and σ do not depend on inter-transmission time distributions. However, notice that the average time for gossip to end, τ, *does* depend on the distributions since time between n^{th} and $(n+1)^{\text{th}}$ events depends on inter-transmission time distributions. Therefore, the following sections would be devoted to obtaining τ although the asymptotic methods we use also provide μ and σ. Moreover, in addition to performance measures, the asymptotic method enables us to see the dynamics of the system, i.e. transient behavior of the system. Although this paper does not aim at the analysis of transient behavior, we will also provide it briefly and compare it with simulation results in Section VI-B. Before moving to the next section, we need to change our settings slightly. Asymptotic results are obtained by taking limit to the number of nodes, i.e. $N \to \infty$. If we start with only one node, initial fraction of nodes having the gossip becomes zero, and the limit process would become degenerate. To prevent this degenerate case, we change the initial condition to be a fixed fraction of nodes, i.e. the fraction of nodes having the gossip at time 0 is αN, for $0 < \alpha < 1$. Since we consider some fraction of nodes instead of exactly one node in the asymptotic regime, one might think that the problem we consider changes significantly. However, we use α just for the derivation of fluid and diffusion models. When we approximate a network having N nodes, by setting $\alpha = 1/N$, we can come back to the problem starting with exactly one node.

In Section V-B, we explain fluid and diffusion models to obtain performance measures under time-varying transmission rates. Using the results, we show some numerical results compared with simulation in Section V-C.

B. Fluid and Diffusion Models

Let $X_{n,t} = (x_{n,t}, y_{n,t})'$ denote the state of the system at time t when population size is $n+1$ where $x_{n,t}$ is the number of nodes actively spreading the gossip and $y_{n,t}$ is the number of nodes not having the gossip by time t. Define λ_t to be the positive deterministic function of t that represents a time-varying transmission rate satisfying $\lambda_t < \infty$ for all t. Then, following the notation in [24], [25], [26], [27], $X_{n,t}$ is the solution to the following integral equation.

$$\begin{aligned} X_{n,t} =& X_{n,0} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} Y_1\Big(\int_0^t \frac{y_{n,s}}{n}\lambda_s x_{n,s} ds\Big) \\ &+ \begin{pmatrix} -1 \\ 0 \end{pmatrix} Y_2\Big(\int_0^t \frac{n - y_{n,s}}{n}\lambda_s x_{n,s} ds\Big), \end{aligned} \qquad (1)$$

where $Y_i(\cdot)$'s are independent Poisson processes with rate 1, $X_{n,0} = (\alpha(n+1), (1-\alpha)(n+1))'$, and α is the fraction of nodes having the gossip at time 0.

Explaining equation (1), the first term in the right hand side is the initial value of the system state: i.e., there are total $\alpha(n+1)$ number of nodes having the information initially. The second term ($Y_1(\cdot)$) represents a counting process where an actively spreading node selects a node not having the information. In this case, $x_{n,t}$ increases by 1, and $y_{n,t}$ decreases by 1, hence $(1,-1)'$. The third term ($Y_2(\cdot)$) counts the number of events when a spreading node chooses a node already having the information, which causes $x_{n,t}$ to decrease by 1, hence $(-1,0)'$.

Note that we can achieve time-varying transmission rates by defining λ_t, and $X_{n,t}$ is the exactly same process considered in Section IV when $\lambda_t = 1$, $\alpha = 1/(n+1)$, and $n = N-1$. Let $\tilde{X}_{n,t} = X_{n,t}/n$. Then, equation (1) is rewritten as follows:

$$\begin{aligned}\tilde{X}_{n,t} =& \tilde{X}_{n,0} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{n} Y_1\Big(n \int_0^t \lambda_s \tilde{y}_{n,s}\tilde{x}_{n,s} ds\Big) \\ &+ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{1}{n} Y_2\Big(n \int_0^t \lambda_s (1-\tilde{y}_{n,s})\tilde{x}_{n,s} ds\Big). \quad (2)\end{aligned}$$

With equation (2), we can obtain its fluid model. All the convergence in this paper is with respect to uniform topology in space D ([1], [2]).

Theorem 6 (Fluid model): Suppose $\{\tilde{X}_{n,t}\}_{n\geq 1}$ is the solution to equation (2). Then, $\tilde{X}_{n,t}$ converges to $\bar{X}_t$ almost surely where $\bar{X}_t$ is the solution to the following integral equation:

$$\begin{aligned}\bar{X}_t &= \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \int_0^t \lambda_s \bar{y}_s \bar{x}_s ds \\ &\quad + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \int_0^t \lambda_s (1-\bar{y}_s)\bar{x}_s ds. \quad (3)\end{aligned}$$

Proof: For $X = (x,y) \in [0,1]\times[0,1]$, $t \leq T$ and $T < \infty$, define $F_t(X)$ to be

$$F_t(X) = \begin{pmatrix} \lambda_t x\big(y-(1-y)\big) \\ -\lambda_t xy \end{pmatrix} = \begin{pmatrix} \lambda_t x(2y-1) \\ -\lambda_t xy \end{pmatrix}.$$

Since $F_t(X)$ is continuous and differentiable on a compact set, it is uniformly continuous and hence there exists a constant M such that

$$|F_t(X) - F_t(Y)| \leq M|X-Y|.$$

By Theorem 2.1 (page 456) in [25], we prove this theorem. ■

Once we solve the integral equation (3), we can approximate $E[\tilde{X}_{n,t}]$ with $\bar{X}_t$ for sufficiently large n. However, in addition to $E[\tilde{X}_{n,t}]$, the fluid model also provides a good way to approximate τ. Define μ_n, σ_n and τ_n to be the mean and standard deviation of reach and the average time elapsed to stop spreading respectively when $n = N-1$.

Theorem 7: For $\beta > 0$, let

$$\gamma_n = \inf\{t : \tilde{x}_{n,t} < \beta\}.$$

Then, γ_n is a stopping time. Define

$$\gamma = \inf\{t : \bar{x}_t < \beta\}.$$

Then, $\lim_{n\to\infty} \gamma_n = \gamma$ almost surely.

Proof: Direct application of the fact that $\tilde{X}_{n,t}$ converges to $\bar{X}_t$ almost surely on $t \in [0,T]$ for some $T < \infty$. ■

Note that γ is a deterministic quantity and can be used for the approximation of $\tau_n = E[\gamma_n]$ when n is large enough.

Although we show in Section V-A that μ and σ are invariant, by utilizing the diffusion model, we can obtain approximations of them. The benefits of using the fluid and diffusion models are its computational scalability and accuracy. When n is small, we can obtain the exact solution of μ and σ using CTMC in Section IV. However, if n becomes large, then it is computationally impossible to calculate them. The fluid and diffusion models provide asymptotic behavior of the system, i.e. they are asymptotically true. Therefore, as n gets large, the approximations become more accurate. Now, we have the fluid model and hence move to the diffusion model.

Theorem 8 (Diffusion model): Let $D_{n,t} = \sqrt{n}(\tilde{X}_{n,t} - \bar{X}_t)$. Define

$$\begin{aligned}K_t &= \lambda_t \begin{pmatrix} 2\bar{y}_t - 1 & 2\bar{x}_t \\ -\bar{y}_t & -\bar{x}_t \end{pmatrix}, \\ L_t &= \sqrt{\lambda_t} \begin{pmatrix} \sqrt{\bar{x}_t\bar{y}_t} & -\sqrt{\bar{x}_t(1-\bar{y}_t)} \\ -\sqrt{\bar{x}_t\bar{y}_t} & 0 \end{pmatrix}.\end{aligned}$$

Then, $D_{n,t}$ converges to D_t in distribution, and D_t is the solution to the following integral equation:

$$D_t = \int_0^t L_s \begin{pmatrix} dW_{1,s} \\ dW_{2,s} \end{pmatrix} + \int_0^t K_s D_s ds,$$

where $W_{i,t}$'s are independent standard Brownian motions.

Proof: The fact that $F_t(X)$ is continuous and twice differentiable on a compact set gives this theorem. See Theorem 3.1 (page 460) and 3.2 (page 463) in [25]. ■

Now, we have the diffusion model and can obtain covariance matrix for D_t using the following theorem.

Theorem 9 (Mean and covariance matrix, [40]): Let Y_t be a d-dimensional stochastic process and the solution to the following linear stochastic differential equation:

$$dY_t = A_t Y_t dt + B_t dW_t, \quad Y_0 = 0,$$

where A_t is a $d\times d$ matrix, B_t is a $d \times k$ matrix, and W_t is a k-dimensional standard Brownian motion. Let $M_t = E\big[Y_t\big]$ and $\Sigma_t = Cov\big[Y_t, Y_t\big]$. Then, M_t and Σ_t are the solution to the following ordinary differential equations:

$$\begin{aligned}\frac{d}{dt} M_t &= A_t M_t \\ \frac{d}{dt}\Sigma_t &= A_t\Sigma_t + \Sigma_t A_t' + B_t B_t'.\end{aligned}$$

From Theorems 6, 8, and 9, we now explain the procedure to approximate μ_n, σ_n, and τ_n for a large n. In Theorem 8, we defined $D_{n,t} = \sqrt{n}(\tilde{X}_{n,t} - \bar{X}_t)$ and $D_{n,t}$ converges to D_t as $n \to \infty$. Therefore, for a large n,

$$\tilde{X}_{n,t} \approx \bar{X}_t + \frac{D_t}{\sqrt{n}}. \quad (4)$$

Then, from equation (4), we get

$$\begin{aligned}E[\tilde{X}_{n,t}] &\approx \bar{X}_t \quad (5) \\ Cov[\tilde{X}_{n,t}, \tilde{X}_{n,t}] &\approx \frac{Cov[D_t, D_t]}{n}. \quad (6)\end{aligned}$$

Analytically, when $t \to \infty$, we can obtain μ_n and σ_n from equations (5) and (6) but practically we can calculate them for sufficiently large t which can be found when the solution of equation (3) is close to an equilibrium point, i.e. for $\varepsilon > 0$, $\inf\{t : |d\bar{x}(u)/du_{u=u_0}| < \varepsilon, \forall u_0 \geq t\}$.

Now, we move our attention to τ. Unlike μ and σ, obtaining τ is not straightforward. In fact, the actual stopping time we want is $\tau_n = \inf\{t : \tilde{x}_{n,t} = 0\}$ and is approximated by $\bar{\tau} = \inf\{t : \bar{x}_t = 0\}$. The problem, however, is the fluid model $\bar{x}_t$ never hits zero, i.e. $\bar{\tau} = \infty$ so this stopping time does not provide a good way to approximate τ_n. A way to resolve this degeneration is to utilize Theorem 7. It is not difficult to see that for a fixed n, the following two stopping times are identical:

$$\begin{aligned} \tau_n &= \inf\{t : \tilde{x}_{n,t} = 0\}, \quad \text{and} \\ \gamma_n &= \inf\{t : \tilde{x}_{n,t} < \beta_n\} \quad \text{for } 0 < \beta_n \leq \frac{1}{n}. \end{aligned}$$

Then, by the Theorem 7, we have

$$\gamma^{\beta_n} = \inf\{t : \bar{x}_t < \beta_n\} \quad \text{for } 0 < \beta_n \leq \frac{1}{n}.$$

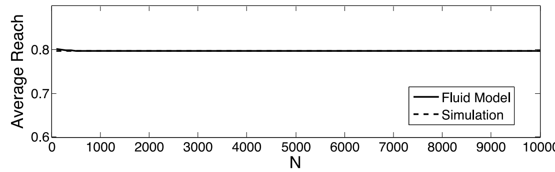
Note that γ_n does not depend on β_n values whereas γ^{β_n} does. Specifically, as $\beta_n \to 0$, $\gamma^{\beta_n} \to \infty$. Thus, if we choose smaller β_n, we are more conservative (larger τ value) and vice versa.

In the following section, we provide several numerical results and compare them with simulation results.

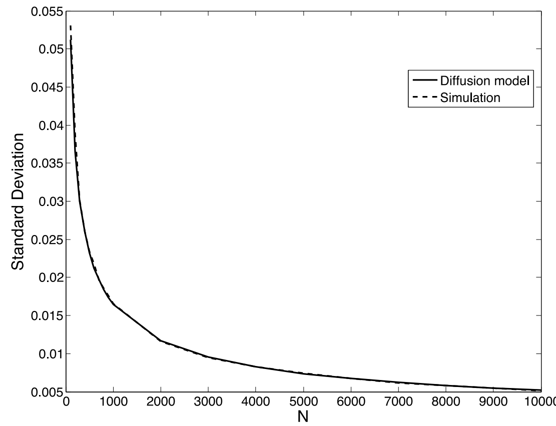
C. Results

We first provide an example having time-varying transmission rates, from which we can observe the invariance of μ and σ described in Theorem 5. For each n, we set the transmission rate is alternating between 1 and 2 every unit time. We use 30,000 simulation runs for each n. The numerical results are obtained from the fluid and diffusion models in Section V-B. From Fig. 7, we notice three interesting things. First, the fluid and diffusion models work great to approximate μ and σ. Second, the invariance of μ and σ in Theorem 5 is numerically and experimentally proven. We have $\mu \approx 79\%$ same as that of CTMC model in Section IV. Although the invariance of σ is not shown explicitly in Fig. 7, we observe same σ values when the fluid model is used for the model in Section IV. Third, the fluid and diffusion models scale extremely well when n gets large. In fact, numerical solutions to them are obtained within a few seconds regardless of the value of n whereas simulation (and CTMC model) time increases according to n.

Now, we move to τ. As described in Section V-B, for a fixed n, different β_n values give different $\tau = \gamma^{\beta_n}$ values and we don't have any rigorous method to pick up the exact value of β_n in this paper. Therefore, we estimate τ by changing β_n in Fig. 8 (a). Empirically, we observe that β_n values between $1/3n$ and $1/2n$ provide reasonably accurate results, and we also verify that smaller β_n values result in more conservative estimation of τ. We left developing a concrete methodology to find appropriate β_n values to future research. However, we suggest a simple informal heuristic to choose β_n values. First, conduct simulation for a small n (but not too small, i.e. $100 \leq n \leq 1000$) which is not computationally expensive. If we

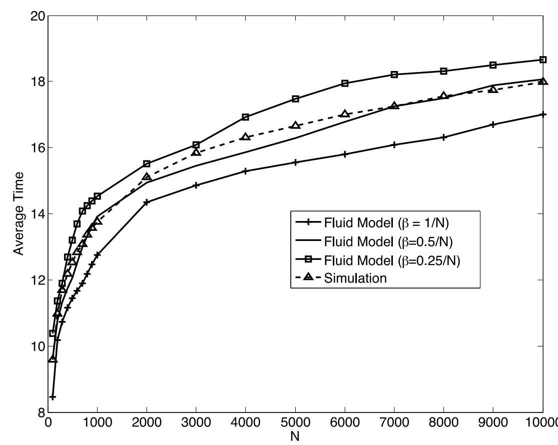


(a) μ vs N

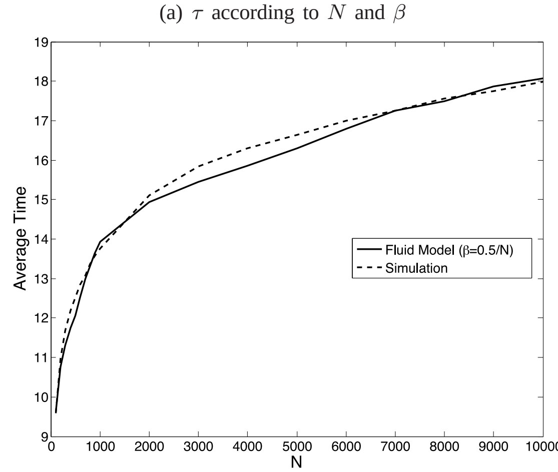


(b) σ vs N

Fig. 7. Comparison of μ and σ between our model and simulation



(a) τ according to N and β



(b) τ when $\beta = 0.5/N$

Fig. 8. τ vs N vs β

consider the CTMC model in Section IV, just use a τ value from the CTMC model for a small n instead of simulation. Second, find β_n value that gives τ the same as that from the first step and use that β_n as a basis to find a constant multiplied to $1/n$. From Fig. 8 (b), we see that the heuristic method achieves reasonably accurate τ.

VI. General inter-transmission time ($F(u) = G(u)$)

In this section, we deal with models having general inter-transmission time distribution. Specifically, we focus on models of which inter-transmission distribution is heavy tailed and has a decreasing pdf described in [33]. But note our proposed method can be applied to more general class of distributions. In Section VI-A, we explain how to obtain a *computationally effective* fluid model and will show some numerical results to verify how our proposed method works when inter-transmission time distribution is heavy tailed.

A. Fluid Model

The main idea of our method is to find a phase-type distribution function which approximates a given distribution function accurately (depending on the problem). For the class of distributions we consider here, i.e. distributions having positive support with decreasing pdfs), the following theorem guarantees that there is a sequence of hyperexponential distributions that converges to an element of that class.

Theorem 10 (Denseness, [33]): If F is a cdf with a completely monotone pdf, then there are hyperexponential cdfs $F^{(n)}, n \geq 1$, i.e., cdfs of the form

$$F^{(n)} = \sum_{i=1}^{k_n} p_{ni}(1 - e^{-\lambda_{ni} t}), \quad t \geq 0,$$

with $\lambda_{ni} \leq \infty$ and $p_{n1} + \cdots + p_{nk_n} = 1$ such that $F^{(n)} \Rightarrow F$ as $n \to \infty$.

Therefore, from Theorem 10, we can choose a hyperexponential distribution function satisfying a desired error bound. After choosing such a hyperexponential distribution function, we combine it with the asymptotic method explained in Section V. In fact, approximating a distribution with phase-type distributions is well studied in the literature. However, one crucial limitation commonly raised in the literature is that a Markov chain obtained from phase-type distributions becomes intractable as the number of phases increases for better approximation, e.g. in our model, the number of states becomes K^n if K is the number of phases and n is the number of nodes. In our method, however, we found that only $K+1$ ODEs are enough to obtain the fluid model of the system.

Suppose the inter-transmission time distribution function of each node is F and is approximated by hyper-exponential distribution function G where

$$1 - G(u) = \sum_{i=1}^{K} p_i \exp(-\lambda_i u).$$

Note we are not interested in how to find G in this paper. We apply an existing approximation method proposed by [33].

Let $X_{n,t} = (x^1_{n,t}, \ldots, x^K_{n,t}, y_{n,t})$, the state of the system and $H_{n,t}$ is the distribution function of inter-transmission time among all nodes at time t with total $n+1$ nodes, then

$$1 - H_{n,t}(u) = \exp\Big(u \sum_{i=1}^{K} \lambda_i x^i_{n,t}\Big).$$

Now consider a split process of Poisson processes. Probability that next event occurs in $x^i_{n,t}$ is

$$\frac{\lambda_i x^i_{n,t}}{\sum_{j=1}^{K} \lambda_j x^j_{n,t}}.$$

Applying minimum of exponential distributions, rate of event is $\sum_{j=1}^{K} \lambda_j x^j_{n,t}$. Therefore, rate of events of each phase is just $\lambda_i x^i_{n,t}$. Let e_i be $K+1$ dimensional column vector where its i^{th} element is 1 and other elements are 0. Then, $X_{n,t}$ is the solution to the following integral equation.

$$\begin{aligned} X_{n,t} =& X_{n,0} \\ &+ \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K} (e_j + e_k - e_i - e_{K+1}) Y^1_{ijk}\Big(\int_0^t \frac{y_{n,s}}{n} \lambda_i x^i_{n,t} p_j p_k\Big) \\ &- \sum_{i=1}^{K} e_i Y^2_i \Big(\int_0^t \frac{n - y_{n,s}}{n} \lambda_i x^i_{n,t}\Big). \end{aligned} \quad (7)$$

Let $\tilde{X}_{n,t} = X_{n,t}/n$. Then, the equation (7) can be written as follows:

$$\begin{aligned} \tilde{X}_{n,t} =& \tilde{X}_{n,0} \\ &+ \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K} \frac{1}{n}(e_j + e_k - e_i - e_{K+1}) Y^1_{ijk}\Big(n\int_0^t \tilde{y}_{n,s} \lambda_i \tilde{x}^i_{n,t} p_j p_k\Big) \\ &- \sum_{i=1}^{K} \frac{1}{n} e_i Y^2_i \Big(n\int_0^t (1 - \tilde{y}_{n,s}) \lambda_i \tilde{x}^i_{n,t}\Big). \end{aligned} \quad (8)$$

Theorem 11 (Fluid model): Suppose $\{\tilde{X}_{n,t}\}_{n\geq 1}$ is the solution to equation (8) and $\tilde{X}_{n,0}$ converges to $\bar{X}_0$ almost surely. Then, $\tilde{X}_{n,t}$ converges to $\bar{X}_t$ almost surely where $\bar{x}^a_t$ and $\bar{y}_t$, the a^{th} and $(K+1)^{\text{th}}$ components of $\bar{X}_t$ respectively, are the solutions to the following differential equations:

$$\begin{aligned} \frac{d}{dt}\bar{x}^a_t =& \bar{y}_t \Big[2p_a \sum_{i=1, i\neq a}^{K} \lambda_i \bar{x}^i_t + (p_a^2 - (1-p_a)^2)\lambda_a \bar{x}^a_t\Big] \\ & -(1-\bar{y}_t)\lambda_a \bar{x}^a_t \quad \text{for } a \in \{1, 2, \ldots, K\}, \\ \frac{d}{dt}\bar{y}_t =& -\bar{y}_t \sum_{i=1}^{K} \lambda_i \bar{x}^i_t \end{aligned}$$

Proof: We will derive the a^{th} component of $\bar{X}_t$. We first consider

$$\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K} \frac{1}{n}(e_j + e_k - e_i - e_{K+1}) Y^1_{ijk}\Big(n\int_0^t \tilde{y}_{n,s} \lambda_i \tilde{x}^i_{n,t} p_j p_k\Big)$$

of equation (8). The a^{th} component, $\bar{x}^a_t$ is

1) increasing by 2 when $j = k = a$ and $i \neq a$ in equation (8) with rate
$$\bar{y}_t p_a^2 \sum_{i=1, i\neq a}^{K} \lambda_i, \bar{x}^i_t.$$
2) increasing by 1
    - when $j = k = i = a$ with rate $\bar{y}_t \lambda_a \bar{x}^a_t p_a^2$.

- when $j = a, k \neq a$, and $i \neq a$ with rate

$$\bar{y}_t \sum_{i=1, i\neq a}^{K} \sum_{k=1, k\neq a}^{K} \lambda_i \bar{x}_t^i p_a p_k = \bar{y}_t \sum_{i=1, i\neq a}^{K} \lambda_i \bar{x}_t^i p_a (1-p_a).$$

- when $k = a, j \neq a$, and $i \neq a$ with rate

$$\bar{y}_t \sum_{i=1, i\neq a}^{K} \sum_{j=1, j\neq a}^{K} \lambda_i \bar{x}_t^i p_a p_j = \bar{y}_t \sum_{i=1, i\neq a}^{K} \lambda_i \bar{x}_t^i p_a (1-p_a).$$

3) decreasing by 1 when $j \neq a, k \neq a$, and $i = a$ with rate

$$\bar{y}_t \sum_{j=1, j\neq a}^{K} \sum_{k=1, k\neq a}^{K} \lambda_a \bar{x}_t^a p_j p_k = \bar{y}_t \lambda_a \bar{x}^a (1 - p_a)^2.$$

Note that $\bar{y}_t$ is decreasing by 1 in any case. Second, we consider

$$-\sum_{i=1}^{K} \frac{1}{n} e_i Y_i^2 \Big(n \int_0^t (1 - \tilde{y}_{n,s}) \lambda_i \tilde{x}_{n,t}^i \Big)$$

of equation (8). The a^{th} component, $\bar{x}_t^a$ is decreasing only when $i = a$ with rate $(1 - \bar{y}_t)\lambda_a \bar{x}_t^a$ and $\bar{y}_t$ does not change in all cases. Therefore, adding all rates obtained above gives the fluid model. ■

The number of ODEs in Theorem 11 is just $K + 1$. The resulting phase-type distributions in [33] have about 15 phases and accurately fit the target distribution function. Therefore, we could think that less than 30 phases are enough for approximation. Even if we have more than 30 phases, solving that number of ODEs is not computationally expensive.

With the fluid model above, we define a stopping time for $\tilde{X}_{n,t}$ and obtain its limit from the following theorem.

Theorem 12: For $\beta \in \mathbb{R}$, let

$$\gamma_n = \inf \Big\{ t : \sum_{i=1}^{K} \tilde{x}_{n,t}^i < \beta \Big\}.$$

Then, γ_n is a stopping time. Define

$$\gamma = \inf \Big\{ t : \sum_{i=1}^{K} \bar{x}_t^i < \beta \Big\}.$$

Then, $\lim_{n\to\infty} \gamma_n = \gamma$ almost surely.

Proof: Since $(\tilde{x}_{n,t}^1, \tilde{x}_{n,t}^2, \ldots, \tilde{x}_{n,t}^K)$ converges to $(\bar{x}_t^1, \bar{x}_t^2, \ldots, \bar{x}_t^K)$ almost surely, by continuous mapping theorem,

$$\sum_{i=1}^{K} \tilde{x}_{n,t}^i \to \sum_{i=1}^{K} \bar{x}_t^i \quad \text{almost surely.}$$

Therefore, by Theorem 7, we prove this theorem. ■

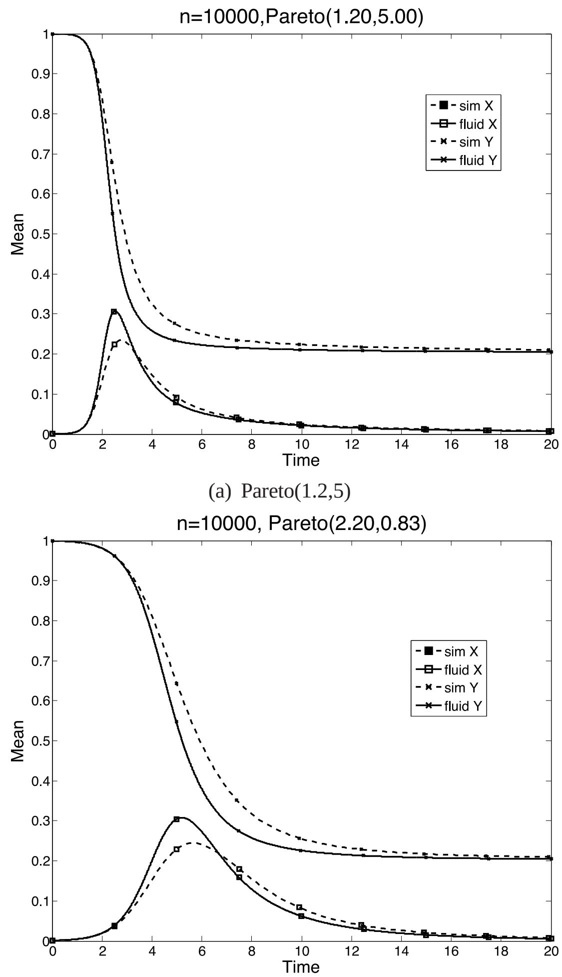
In a similar fashion in Section V-B, by Theorem 12, we have

$$\gamma^{\beta_n} = \inf \Big\{ t : \sum_{i=1}^{K} \bar{x}_t^i < \beta_n \Big\} \quad \text{for } 0 < \beta_n \leq \frac{1}{n},$$

and use it as approximation of τ. In the following section, we provide some numerical results along with simulations.

B. Results

We first provide two examples to illustrate how the fluid model approximates the system when inter-transmission time distributions are Pareto(1.2,5) (mean 1 with infinite variance) and Pareto(2.2,0.83) (mean 1 with finite variance). The notation, Pareto(a,b), is from [33]. Fig. 9 illustrates the average fraction of nodes actively spreading a gossip (denoted by X in figure) and not received it yet (denoted by Y in figure) at time t. Considering computational efficiency of our method, the fluid model shows reasonable accuracy as seen in Fig. 9. From the figure, we also observe the invariance of μ in Theorem 5, i.e. the fluid models in both cases eventually end up with the same μ value ($\approx 79\%$).

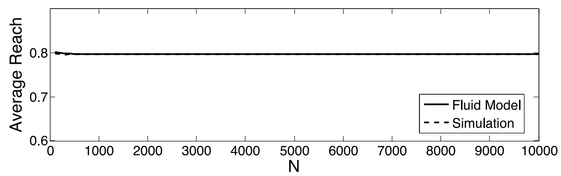


(a) Pareto(1.2,5)

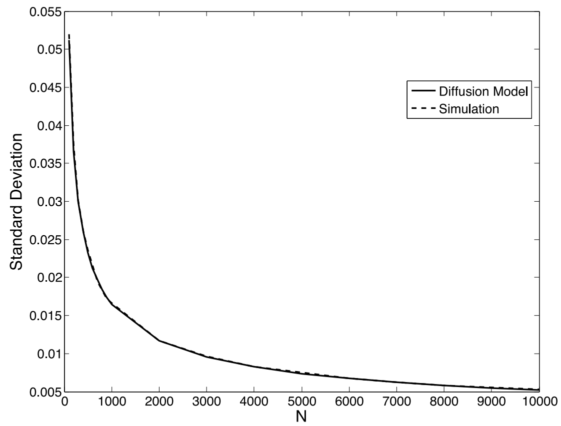
(b) Pareto(2.2,0.83)

Fig. 9. Fluid Model vs Simulation

We also verify the invariance of μ and σ from Fig. 10. Note in Fig. 10, the fluid and diffusion models to obtain μ and σ are those for time-varying transmission rates and the simulation result is actually from the Pareto inter-transmission time distribution. Finally, Fig. 11 shows estimation of τ according to the population size and β values. From the figure, we also observe that smaller β values give more conservative estimation of τ vice versa and our heuristic described in Section V-C still provides reasonable accuracy.

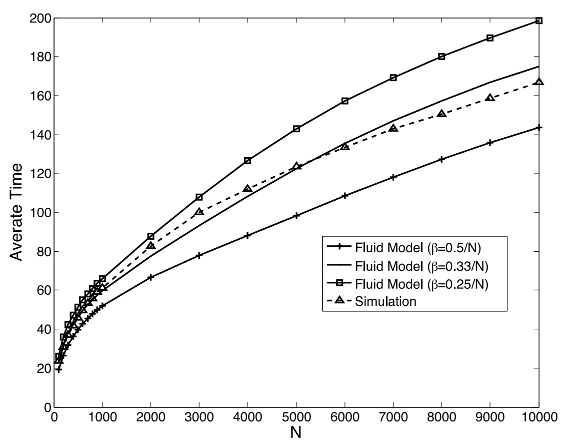


(a) μ vs N

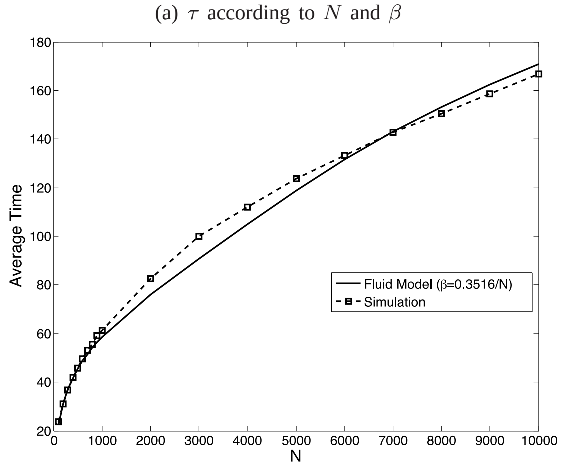


(b) σ vs N

Fig. 10. Comparison of μ and σ between our model and simulation



(a) τ according to N and β



(b) τ when $\beta = 0.3516/N$

Fig. 11. τ vs N vs β

However, choosing an appropriate β_n value is left for future research.

VII. Conclusion

In this paper we propose an epidemic-based information dissemination model (with explicit stopping criterion) to route information to as many nodes as possible under harsh environmental conditions. In the literature typically the models considered are analyzed using simulations, however we use an analytical model based on Markov chains to obtain performance measures that would facilitate quick what-if analysis and evaluate design alternatives. Also, most studies in the literature do not consider stopping criteria explicitly. Besides the information spreading models and methods of analyses, another contribution of this paper is the curious finding that the average fraction of nodes that eventually receive the routed information is a parameter-free constant (even for reasonably small N, the total number of sensor nodes). In addition, we also illustrate that there is an insignificant difference in performance between the synchronous and asynchronous time information transmission models. Therefore it does not appear worth it to synchronize which is a useful insight, although several articles in the literature do consider the synchronous case. To make our algorithm more realistic, we extended the asynchronous model in two ways, i.e. time-varying transmission rates and general inter-transmission time distributions.

For time-varying transmission rates, we utilized fluid and diffusion approximations and found that they provided not only accurate results but also computational efficiency regardless of the number of nodes. Considering general inter-transmission time distributions was more challenging. We used phase-type approximation of a general distribution and combined it with the fluid model. It turns out that the fluid model contains only $K+1$ number of ODEs that need to be solved where K is the number of phases and is fairly scalable.

There are a few limitations for this paper. First of all, the assumptions for the analysis of the algorithm are accurate only under the framework of the setting mentioned in the paper: nodes move rapidly and only rarely they are in a transmission phase (under other conditions the analysis would be an approximation and we need to investigate the performance); in addition, there is a requirement that it is enough if many (but not all) nodes receive the information which is reasonable for many applications. However, it is worthwhile to point out that it is fairly straightforward to modify our methodology to say different stopping criteria, or non-zero transmission time, modifications to the gossip protocol. etc. The main concern is that the notation would become more cumbersome. In addition, further research is required to find a way to choose an appropriate β value which does affect the estimation accuracy.

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