A Fluid Queueing Model for Link Travel Time Moments[1]

Jeffrey P. Kharoufeh[2]
Department of Operational Sciences
Air Force Institute of Technology
AFIT/ENS, Building 640
2950 Hobson Way
Wright Patterson AFB, OH 45433-7765 USA
and
Natarajan Gautam
Harold and Inge Marcus Department of Industrial
and Manufacturing Engineering
The Pennsylvania State University
310 Leonhard Building
University Park, PA 16802 U.S.A.
Email: ngautam@psu.edu

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Abstract

We analyze the moments of the random time required for a vehicle to traverse a transportation network link of arbitrary length when the vehicle's speed is governed by a random environment. The problem is motivated by stochastic transportation network applications in which the estimation of travel time moments is of great importance. We analyze this random time in a transient and asymptotic sense by employing techniques from the field of fluid queues. The main results are demonstrated on two example problems.

Keywords: Link travel time; random environment; stochastic modeling

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[2]Corresponding author, Ph: (937) 255-6565 x 4336; Fax: (937) 656-4943; Email: Jeffrey.Kharoufeh@afit.edu

1 Introduction

We consider the movement of a vehicle along a transportation network link (such as a freeway segment) whose length is x units. The vehicle's speed varies over the course of its sojourn due to the influence of an underlying environment process. Owing to the effects of this random environment, the time required to traverse the link is a random variable denoted by $T(x)$. In this paper, our aim is to analyze the moments of the random travel time $T(x)$, in the transient and the asymptotic sense, with the ultimate objective of obtaining computationally expedient measures that are extremely useful in a number of transportation contexts. To that end, we demonstrate that the problem can be viewed and analyzed as a fluid queueing model from which such measures may be derived. The approach allows us to consider the travel time moments for an individual link of a transportation network subject to a randomly evolving environment. This random environment is characterized as a continuous-time stochastic process on a finite sample space.

The dynamics of the model can be described as follows. If at time $t \geq 0$ the underlying finite-state process denoted by $\{Z(t) : t \geq 0\}$ is in state $i \in S \equiv \{1, 2, \ldots, K\}$, then the speed of the vehicle is a strictly positive quantity V_i. The stochastic process $\{Z(t) : t \geq 0\}$ is the random environment process. If $D(t)$ denotes the cumulative distance travelled by the vehicle up to time $t > 0$, and $\{Z(t) : t \geq 0\}$ is a Markov process, then it can be shown that $\{(D(t), Z(t)) : t \geq 0\}$ is also a Markov process. Moreover, the random variable $D(t)$ is an additive functional of $Z(t)$. The process $\{(D(t), Z(t)) : t \geq 0\}$ is used to analyze the moments of the random time $T(x)$ which corresponds to a transient first passage time for the process $\{D(t) : t \geq 0\}$.

Our results are primarily motivated by problems in transportation and logistics where it is important for decision makers to know the travel time moments of individual vehicles in a stochastic transportation network. The main contribution of the work is the novel application of fluid queueing techniques for the analysis of an individual vehicle whose time-variant speed is modulated by a random environment. This is in contrast to an aggregated

approach for all vehicles usually found in the traffic flow theory (cf. Lighthill and Whitham [12] and Highway Capacity Manual [7]). Our approach is intuitive in the sense that a plot of the vehicle's speed over time corresponds to a sample path of a continuous-time stochastic process, possibly on a finite state space. Owing to its generality, the model may be used in a number of transportation settings directly or by making suitable alterations. We describe in detail applications in ground and maritime transportation.

- *Ground transportation*: Consider a vehicle that traverses a roadway segment with x corresponding to the physical length of the segment. Several factors influence the speed of the vehicle as it attempts to traverse the roadway segment. Some of those may be physical factors (e.g., roadway geometry, grades, visibility), traffic factors (e.g., density, presence of heavy vehicles, merging traffic), or environmental factors (e.g., weather conditions, speed limits, etc.). It is assumed that the environment process $\{Z(t) : t \geq 0\}$ is known, and thus, may be used to obtain the moments of the random variable $T(x)$. These moments can then be applied to construct parametric distributions for stochastic arc weights in transportation networks within the context of automatic route guidance systems described in [6] or in least-time stochastic transportation network problems such as those described in [13] and [14].

- *Maritime transportation*: The environment process approach is similarly applicable in maritime scenarios. In particular, consider a ship traversing one leg of its journey of length x. Stochastic and dynamic weather conditions directly influence the speed with which the ship may travel. In such case, the stochastic process $\{Z(t) : t \geq 0\}$ may be used to model the set of meteorological variables which determines the ship's speed at a given point in time (and possibly space). By assigning a cost for each speed (e.g., fuel consumption) and making appropriate alterations, our model may be used to compute the expected cost incurred for traversing each leg of the ship's sojourn. Furthermore, if each arc of a network is governed by its own environment process, then it may be possible to solve a stochastic and dynamic minimum cost problem such as

that considered by Psaraftis and Tsitsiklis [17].

Throughout the remainder of this paper, we concern ourselves with the general setting of a vehicle traversing a link of length x with the understanding that this may pertain to either of the above scenarios. Moreover, it is possible to model a number of real-world contexts with appropriate alterations to the problem parameters and their physical interpretations.

The main contributions of this work can be summarized as follows. First, using the tools of fluid queueing models, we present a transparent approach for implicitly incorporating the time dependence of speed for a vehicle traversing a link of length x. Next, we give an explicit matrix transform expression for the rth moment of the link travel time which gives exact results when the transform can be algebraically inverted, and very accurate approximate results with numerical transform inversion. Third, we discuss asymptotic results for the first and second moments of the link travel time which serve as computationally expedient approximations for surrogate link travel time distribution parameters.

The remainder of the paper is organized as follows. The next section reviews the pertinent concepts from the theory of fluid queues and demonstrates the means by which the theory is applied to the link travel time problem. Section 3 demonstrates how to compute the moments of the link travel time for a link of arbitrary (but finite) length x. In section 4, we use the transform results of section 3 to provide intuitive asymptotic expressions for the mean and variance of the link travel time. Section 5 presents numerical results on two example problems followed by our concluding remarks in section 6.

2 Mathematical Model

2.1 Fluid Queueing Concepts

In this section, we provide a brief overview of the notation and rudimentary concepts of fluid queueing models. A fluid queueing model can be described as one in which the input to a stochastic system is modelled as a continuous fluid that enters a buffer and then leaves the buffer through an output channel (service mechanism) with constant output capacity c.

One measure of importance for such systems is the amount of fluid contained in the buffer at time t denoted by the random variable $X(t)$. The stochastic process, $\{X(t) : t \geq 0\}$, is often referred to as the buffer content process. There exists an external process called the random environment process that modulates the input of fluid to the buffer. That is, the state of the random environment process dictates the rate at which fluid flows into the buffer.

Suppose $\{Z(t) : t \geq 0\}$ denotes the random environment process that drives fluid generation. Define $R_{Z(t)}$ as the entrance rate of fluid to the buffer at time t and let the drift function of this process be

$$\phi_{Z(t)} \equiv R_{Z(t)} - c. \tag{1}$$

The overall storage capacity of the buffer is denoted by a fixed, deterministic value B. When $B = \infty$, the dynamics of the buffer-content process are given by

$$\frac{dX(t)}{dt} = \begin{cases} \phi_{Z(t)} & X(t) > 0 \\ \phi^+_{Z(t)} & X(t) = 0 \end{cases}$$

where $w^+ \equiv \max\{w, 0\}$. In case $B < \infty$, the system is governed by

$$\frac{dX(t)}{dt} = \begin{cases} \phi^+_{Z(t)} & X(t) = 0 \\ \phi_{Z(t)} & 0 < X(t) < B \\ -\phi^-_{Z(t)} & X(t) = B \end{cases}$$

where $w^- \equiv \max\{0, -w\}$. The probability law of the buffer-content process, $\{X(t) : t \geq 0\}$ is dictated by the form of the random environment process, $\{Z(t) : t \geq 0\}$, and the associated function ϕ. This function, referred to as the drift function, corresponds to the net input rate of fluid to the buffer (entrance rate - exit rate). The diagonal matrix $\Phi \equiv diag(\phi_1, \phi_2, \ldots, \phi_K)$ is called the drift matrix. The instant at which the fluid level crosses some fixed level x corresponds directly to a first passage time for the stochastic process $\{X(t) : t \geq 0\}$.

Many researchers in the field of telecommunications have recognized the utility of such models for solving engineering problems. A few important papers in this area are due to Antick, *et al.* [2], Elwalid and Mitra [5], Kesidis, *et al.*, [8] and Kulkarni and Gautam [11]. Other researchers have considered more generalized fluid queueing problems and some

good examples are the papers due to Asmussen [3], Rogers [18], and more recently, Takada [19]. The approach of our paper is to employ a fluid queueing approach for the mathematical characterization of link travel time moments as simple expressions that may be computed in a computationally expedient manner. We next demonstrate the means by which this may be accomplished.

2.2 Fluid Model for Vehicle Displacement

Consider a vehicle that must traverse a link of length x whose speed is governed by a random environment on a finite state space $S = \{1, 2, \ldots, K\}$ where $K \in \mathbb{N}$, $K \geq 2$. Define the random environment process by $\{Z(t) : t \geq 0\}$ so that at time t, the vehicle assumes a (strictly positive) speed $V_{Z(t)}$. Since the environment process has a finite state space, the vehicle may assume speeds in the finite set $\{V_i : i = 1, 2, \ldots, K\}$, where $V_i > 0$ for all i. Moreover, define the matrix $V = diag(V_1, V_2, \ldots, V_K)$. Hence, the time dependence of vehicle speed is captured implicitly through the environment process, $\{Z(t) : t \geq 0\}$. The initial conditions experienced by the vehicle as it begins its sojourn of the link are captured by the initial state of the random environment, and we denote the initial distribution of the environment process by a row vector, $z_0 = [P\{Z(0) = i\}]$. Define by $D(t)$, the total displacement of the vehicle up to time t and the associated stochastic process $\{D(t) : t \geq 0\}$. With these definitions, we next demonstrate that the link travel time model is analogous to a fluid queueing model. In order to make this point more lucid, Table 1 summarizes the relationship between the concepts of a fluid queueing model and those of the link travel time model.

Table 1: Analogous fluid queueing and transportation concepts.

Fluid Queueing Concept	Transportation Analogy
Environment process Z	Environment process Z
Drift function ϕ_i, $i \in S$	Velocity function V_i, $i \in S$
Drift matrix Φ	Velocity matrix V
Buffer content process X	Vehicle displacement process D
Fluid level crossing time $T(x)$	Vehicle distance crossing time $T(x)$

Though many analogous concepts exist between the two models, there also exist some noteworthy distinctions. In our model, we limit the movement of a vehicle to the positive direction and allow only positive velocities. Once the vehicle begins its sojourn, it does not stop, nor does it move in the negative direction. The analogous situation in the fluid queueing model is that the output capacity of the system is zero ($c = 0$), and the buffer accumulates fluid until it first reaches the threshold value x. Hence, our link travel time model is a special case of a general fluid queueing model in which the drift rates (ϕ_i, $i = 1, 2, \ldots, K$) are all positive, and the D process, corresponding to the buffer content process X, possesses monotonically increasing sample paths. The cumulative distance travelled by the vehicle up to time $t > 0$ is defined by

$$D(t) = \int_0^t V_{Z(u)} du. \tag{2}$$

Equation (2) indicates that D is a Markov additive functional of Z, and thus, the random time required to first traverse a link of length x is the first passage time

$$T(x) = \inf\{t : D(t) > x\}. \tag{3}$$

When the process $\{Z(t) : t \geq 0\}$ is an irreducible, continuous-time Markov chain (CTMC) on $S = \{1, 2, ..., K\}$ with infinitesimal generator matrix $Q = [q_{ij}]$, the dynamics of the buffer-content process are well understood. For example, Rogers [18] provided the stationary probability law of the buffer-content process when the buffer is finite or infinite using the Wiener-Hopf factorization for the generator matrix of the governing Markov process. Asmussen [3] derives the stationary distribution of the same process in two cases: with

and without an additional Brownian component. Recently, Takada [19] presented a fluid queueing model that generalizes previous results for the MAP/G/1 queue by allowing the buffer-content process to additionally include jumps. These works provide mathematically elegant approaches for computing the time-stationary probability distribution of the buffer-content process. By contrast, the link travel time problem requires the transient probability law for the D process as indicated by Equation (3). For the purposes of this work, we consider a transform approach for the transient probability distribution of $D(t)$ from which we obtain (spatially) transient and asymptotic measures for the link travel time moments. The following review of this transient distribution follows from [9].

Define the joint distribution function

$$H_i(x,t) = P\{D(t) \leq x, Z(t) = i\}, \qquad i \in S. \tag{4}$$

Moreover, define the cumulative distribution function for $T(x)$ by

$$\begin{aligned} G(x,t) \equiv P\{T(x) \leq t\} &= 1 - P\{D(t) \leq x\} \\ &= 1 - \sum_{i \in S} H_i(x,t). \end{aligned} \tag{5}$$

Let $H(x,t) = [H_i(x,t)]_{i \in S}$ denote a $1 \times K$ vector. It can be shown that the joint probability distribution $H(x,t)$ satisfies the partial differential equation (PDE),

$$\frac{\partial H(x,t)}{\partial t} + \frac{\partial H(x,t)}{\partial x} V = H(x,t)Q. \tag{6}$$

with initial condition $H(x,0) = z_0$. Denote by $H^*(x,s_2)$, the Laplace transform (LT) of $H(x,t)$ with respect to t given by

$$H^*(x,s_2) = \int_0^\infty e^{-s_2 t} H(x,t) dt, \tag{7}$$

and the Laplace-Stieltjes transform (LST) of $H^*(x,s_2)$ with respect to x as

$$\tilde{H}^*(s_1,s_2) = \int_0^\infty e^{-s_1 x} dH^*(x,s_2). \tag{8}$$

so that $\tilde{H}^*(s_1,s_2) = [\tilde{H}_i^*(s_1,s_2)]_{i \in S}$ is a $1 \times K$ row vector of transform expressions. The following result of [9] can be used in conjunction with Equation (5) to obtain the probability distribution of the random link travel time $T(x)$.

Theorem 1 *The solution to the partial differential equation (6) in the transform space is given by*

$$\tilde{H}^*(s_1, s_2) = z_0(s_1 V + s_2 I - Q)^{-1} \tag{9}$$

where $\tilde{H}^*(s_1, s_2)$ *is defined by Equation (8),* z_0 *is the initial environment distribution, and* s_1 *and* s_2 *are complex transform variables with* $Re(s_1) > 0$ *and* $Re(s_2) > 0$.

Finally, it is easy to see that the cumulative distribution function for $T(x)$ is given by

$$\tilde{G}^*(s_1, s_2) = \frac{1}{s_2} - z_0 \tilde{H}^*(s_1, s_2) e. \tag{10}$$

where e denotes a K-dimensional column vector of ones. It should be noted that the LST of the link travel time distribution can be obtained as a one-dimensional transform with respect to the temporal variable t in a slightly more mathematically elegant manner. In particular, we note that our problem is a special case of the problems considered by both Rogers [18] and Asmussen [3] with the exception that Equation (9) corresponds to a transient distribution.

The distributions of this section can be accurately computed using a two-dimensional numerical inversion algorithm such as the one due to Moorthy [15]. However, it is possible to generate approximate distributions with far less computational effort by computing the moments of the link travel time and using them in surrogate, parametric distributions. This approach may be especially useful in the analysis of stochastic transportation networks wherein the entire cumulative distribution function is needed to compute stochastically shortest paths. However, in lieu of the link travel time distribution, the moments of this random time can be computed in a simple fashion. Moreover, asymptotic approximations of the link travel time moments can be obtained as closed-form analytical expressions. In section 3, we show how to compute the moments of the random link travel time using Equation (9) when the length of the link (x) is finite.

3 Transient Link Travel Time Moments

In this section, we derive an expression using the fluid queueing approach for the moments of the link travel time whenever the link length is finite. By Equation (5),

$$G(x,t) = 1 - \sum_{i \in S} H_i(x,t),$$

where G is the CDF of the random link travel time, $H_i(x,t) = P\{D(t) \leq x, Z(t) = i\}$, and S is the finite state space of the random environment process $\{Z(t) : t \geq 0\}$ that modulates vehicle speed. It is well-known that the rth moment of the random variable $T(x)$ may be obtained by evaluating at 0, the rth-order derivative of the Laplace-Stieltjes transform (LST) of G which is given by

$$\tilde{G}(x,s_2) = \int_0^\infty e^{-s_2 t} dG(x,t) = 1 - \sum_{i \in S} H_i^*(x,s_2)s_2, \tag{12}$$

where $H_i^*(x,s_2)$ is given by Equation (7). The rth moment of the link travel time, denoted by $m_r(x)$, is

$$m_r(x) \equiv E[(T(x))^r] = (-1)^r \frac{\partial^r}{\partial s_2^r} \tilde{G}(x,s_2)\bigg|_{s_2=0.} \tag{13}$$

Next define

$$\begin{aligned} K_{s_2}^r(x) &= (-1)^r \frac{\partial^r \tilde{G}(x,s_2)}{\partial s_2^r} \\ &= (-1)^{r+1} \left(s_2 \sum_{i \in S} \frac{\partial^r H_i^*(x,s_2)}{\partial s_2^r} + r \sum_{i \in S} \frac{\partial^{r-1} H_i^*(x,s_2)}{\partial s_2^{r-1}} \right). \end{aligned} \tag{14}$$

Equation (14), which is derived from Equation (12), implies that

$$m_r(x) = K_0^r(x) = (-1)^{r+1} r \sum_{i \in S} \frac{\partial^{r-1} H_i^*(x,s_2)}{\partial s_2^{r-1}}\bigg|_{s_2=0.} \tag{15}$$

In order to solve the differential Equation (15), transform methods are again employed. The LST of $m_r(x)$ with respect to x is

$$\tilde{m}_r(s_1) = \int_0^\infty e^{-s_1 x} dm_r(x).$$

By taking the LST of Equation (15) on both sides,

$$\begin{aligned}\tilde{m}_r(s_1) &= (-1)^{r+1} r \sum_{i \in S} \left. \frac{\partial^{r-1} \tilde{H}_i^*(s_1, s_2)}{\partial s_2^{r-1}} \right|_{s_2=0} \\ &= (-1)^{r+1} r \left. \frac{\partial^{r-1} \tilde{H}^*(s_1, s_2)}{\partial s_2^{r-1}} \right|_{s_2=0} e \end{aligned} \tag{16}$$

where $\tilde{H}^*(s_1, s_2)$ is the matrix transform of Equation (9). Assuming the existence of all derivatives of $H_i(x, s_2)$ at $s_2 = 0$, inversion of Equation (16) yields the rth moment of the random link travel time. The following lemma will be needed to derive a matrix expression for Equation (16).

Lemma 1 *The kth order partial derivative of the vector $\tilde{H}^*(s_1, s_2)$ with respect to s_2 is*

$$\frac{\partial^k \tilde{H}^*(s_1, s_2)}{\partial s_2^k} = (-1)^k k! z_0 \left(s_1 V + s_2 I - Q\right)^{-k-1}, \quad k \geq 0.$$

Lemma 1 can be easily proved by mathematical induction and is next used to derive a general expression for the rth moment of the link travel time.

Theorem 2 *The Laplace-Stieltjes transform of $m_r(x)$ is given by*

$$\tilde{m}_r(s_1) = r! z_0 (s_1 V - Q)^{-r} e. \tag{17}$$

Proof. Applying Lemma 1 to Equation (16) directly shows that

$$\begin{aligned}\tilde{m}_r(s_1) &= (-1)^{r+1} r \left. \frac{\partial^{r-1}}{\partial s_2^{r-1}} \tilde{H}^*(s_1, s_2) \right|_{s_2=0} e \\ &= (-1)^{r+1} r (-1)^{r-1} (r-1)! z_0 \left[(s_1 V - Q)^{-1}\right]^r e \\ &= r! z_0 (s_1 V - Q)^{-r} e, \end{aligned}$$

and the proof is complete. ∎

Equation (17) gives an exact analytical expression for the LST of the rth moment of the random link travel time, provided that all derivatives exist at $s_2 = 0$. In some cases, the transform may be inverted algebraically for an exact solution. However, very close approximations may be obtained via numerical inversion in only one dimension by using a number of widely available inversion algorithms such as the one due to Abate and Whitt [1].

4 Asymptotic Link Travel Time Moments

In this section, the asymptotic behavior of the moments of the random link travel time is considered. In subsections 4.1 and 4.2, we investigate the first and second moments of the link travel time in the asymptotic region (as $x \to \infty$) when the speed is modulated by a CTMC, $\{Z(t) : t \geq 0\}$. In section 4.3, the asymptotic variance is considered. Asymptotic approximations often yield computationally expedient expressions that can drastically reduce computational effort by eliminating the need for iterative algorithms. Furthermore, these approximations may be used to construct surrogate, parametric distributions for the link travel time.

4.1 Asymptotic First Moment of $T(x)$

In order to prove our result for the asymptotic mean of the random link travel time, we first need the following lemma.

Lemma 2 *Let Q be the positive recurrent infinitesimal generator matrix for the environment process, $\{Z(t) : t \geq 0\}$, having stationary distribution $p = [p_j]_{j \in S}$. Then the matrix $\hat{Q} = V^{-1}Q$ is an infinitesimal generator for a CTMC, $\{\hat{Z}(t) : t \geq 0\}$, with limiting distribution $\hat{p} = [\hat{p}_j]_{j \in S}$ given by*

$$\hat{p}_j = \frac{p_j V_j}{pv}, \quad j \in S, \tag{18}$$

which satisfies $\hat{p}\hat{Q} = \boldsymbol{0}$ and $\hat{p}e = 1$, where $v = Ve$ and $pv = \sum_j p_j V_j$.

Proof. The proof is immediate since $V^{-1}Q$ is positive recurrent and clearly possesses the stated unique, stationary distribution. ■

Theorem 3 provides an intuitive result for the mean link travel time, namely, that the long-run average link travel time divided by its displacement converges to the reciprocal of the long-run average speed of the vehicle.

Theorem 3 *As* $x \longrightarrow \infty$,

$$\frac{m_1(x)}{x} \longrightarrow \frac{1}{pv}. \tag{19}$$

Proof. By the asymptotic properties of the LST, it is well-known (Kulkarni [10], p. 583) that

$$\lim_{s_1 \to 0} s_1 \tilde{m}_1(s_1) = \lim_{x \to \infty} m_1(x)/x.$$

Thus, it will be shown that $s_1 \tilde{m}_1(s_1) \to (pv)^{-1}$ as $s_1 \to 0$. Applying Theorem 2

$$\begin{aligned} s_1 \tilde{m}_1(s_1) &= s_1 z_0 (s_1 V - Q)^{-1} e \\ &= z_0 s_1 (s_1 I - V^{-1} Q)^{-1} V^{-1} e \end{aligned} \tag{20}$$

By Lemma 2, $\hat{Q} = V^{-1}Q$ is a generator matrix for the CTMC, $\{\hat{Z}(t) : t \geq 0\}$ with probability transition matrix $\hat{P}(t)$ satisfying the forward equation,

$$\frac{d\hat{P}(t)}{dt} = \hat{P}(t)\hat{Q}. \tag{21}$$

Transform methods are employed to solve Equation (21). After simplication, the LST of $\hat{P}(t)$, denoted by, $\Psi(s_1)$, is

$$\Psi(s_1) = s_1 (s_1 I - \hat{Q})^{-1}. \tag{22}$$

By the limiting properties of the LST (Kulkarni [10]),

$$\lim_{s_1 \to 0} \Psi(s_1) = \lim_{t \to \infty} \hat{P}(t) = \hat{P}(\infty), \tag{23}$$

where the j^{th} column of $\hat{P}(\infty)$ has $\hat{p}_j$ of Equation (18) for each row, provided the CTMC is ergodic. Now, substituting Equations (22) and (17) into Equation (20) gives

$$\begin{aligned} \lim_{s_1 \to 0} s_1 \tilde{m}_1(s_1) &= \lim_{s_1 \to 0} z_0 s_1 (s_1 I - V^{-1} Q)^{-1} V^{-1} e \\ &= z_0 \hat{P}(\infty) V^{-1} e \\ &= (pv)^{-1} \end{aligned}$$

and the result is obtained. ∎

4.2 Asymptotic Second Moment of $T(x)$

In this subsection, we provide a similar, intuitive result for the asymptotic second moment of the random link travel time. The following lemma is needed to prove the result.

Lemma 3 *Let $f(\cdot)$ be a function of exponential order on the positive real line such that $f(t) \to \infty$ and $f'(t) \to \infty$ as $t \to \infty$. Let the Laplace transform of f be denoted by $f^*(s)$ and let its Laplace-Stieltjes transform be $\tilde{f}(s)$. Then,*

$$\lim_{s\to 0} s^3 f^*(s) = \lim_{s\to 0} s^2 \tilde{f}(s) = 2 \lim_{t\to\infty} \frac{f(t)}{t^2}. \tag{24}$$

Proof. The lemma will be proved by considering the Laplace transform of the third derivative of f. Assuming the existence of this transform, we have that (see Churchill [4])

$$\mathcal{L}\left(\frac{d^3 f}{dt^3}\right) = \int_0^\infty e^{-st} f^{(3)}(t)dt \;\; = \;\; s^3 f^*(s) - s^2 f(0) - s f'(0) - f^{(2)}(0).$$

where $f^{(n)}$ $(n \geq 2)$ denotes the n^{th}-order derivative of f with respect to t. Letting $s \to 0$ on both sides of the above equation yields,

$$\begin{aligned} \lim_{s\to 0} s^3 f^*(s) - f^{(2)}(0) &= \lim_{s\to 0} \int_0^\infty e^{-st} f^{(3)}(t)dt \\ &= \lim_{a\to\infty} \int_0^a f^{(3)}(t)dt \\ &= \lim_{a\to\infty} f^{(2)}(a) - f^{(2)}(0), \end{aligned}$$

which implies,

$$\lim_{s\to 0} s^3 f^*(s) = \lim_{t\to\infty} f^{(2)}(t). \tag{25}$$

It will next be shown that the right-hand side of Equation (24) is equal to the right-hand side of Equation (25).

$$\begin{aligned} 2 \lim_{t\to\infty} \frac{f(t)}{t^2} &= 2 \lim_{t\to\infty} \frac{f'(t)}{2t} \\ &= \lim_{t\to\infty} f^{(2)}(t). \end{aligned}$$

The equality is obtained by applying L'Hospital's rule twice. ∎

Theorem 4 *Assume that* $m_2'(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$. *Then as* $x \longrightarrow \infty$

$$\frac{m_2(x)}{x^2} \longrightarrow \frac{1}{(pv)^2}.$$

Proof. By Lemma 3, it follows directly that

$$\lim_{s_1 \to 0} s_1^2 \tilde{m}_2(s_1) = 2 \lim_{x \to \infty} m_2(x)/x^2.$$

Thus, it will be shown that $s_1^2 \tilde{m}_2(s_1) \to 2(pv)^{-2}$ as $s_1 \to 0$.

$$\begin{aligned} \lim_{s_1 \to 0} s_1^2 \tilde{m}_2(s_1) &= \lim_{s_1 \to 0} 2 z_0 s_1 (s_1 I - V^{-1}Q)^{-1} V^{-1} s_1 (s_1 I - V^{-1}Q)^{-1} V^{-1} e \\ &= 2 z_0 \hat{P}(\infty) V^{-1} \hat{P}(\infty) V^{-1} e \\ &= 2(pv)^{-2} \end{aligned}$$

which implies $m_2(x)/x^2 \to (pv)^{-2}$. ∎

Theorems 3 and 4 indicate that the asymptotic approximation for $m_1(x)$ is $\hat{m}_1(x) = x/pv$ and the asymptotic approximation for $m_2(x)$ is given by $\hat{m}_2(x) = x^2/(pv)^2 = (\hat{m}_1(x))^2$. Thus, the asymptotic approximations for the first two moments of the link travel time reduce to simple functions of the link's length x. Before proceeding to an analysis of the variance of the random variable $T(x)$, it should be noted that the transform expressions of Theorems 3 and 4 directly provide insight to the limiting behavior of $T(x)/x$, as indicated in the following corollary.

Corollary 1 *Assume* $m_2'(x) \longrightarrow \infty$ *as* $x \longrightarrow \infty$. *Then as* $x \longrightarrow \infty$

$$T(x)/x \longrightarrow_P 1/pv,$$

i.e., the random variable $T(x)/x$ *converges to* $1/pv$ *in probability.*

Proof.

$$\lim_{x \to \infty} Var\left(\frac{T(x)}{x}\right) = \lim_{x \to \infty} x^{-2} \left\{ m_2(x) - m_1^2(x) \right\} = 0. \tag{26}$$

Fix $\epsilon > 0$. By Chebyshev's inequality and Equation (26), we have that

$$P\left\{\left|\frac{T(x)}{x} - \frac{1}{pv}\right| \geq \epsilon\right\} \leq \frac{1}{\epsilon^2} Var\left(T(x)/x\right) \longrightarrow 0.$$

■

The corollary states that, just as $m_1(x)/x \to 1/pv$, the random variable $T(x)/x$ converges to the reciprocal of the long-run, average speed, even though $T(x) \to \infty$ as $x \to \infty$. In the following subsection, it will be shown that the standard deviation of the link travel time in the asymptotic regime is proportional to the square root of the link length.

4.3 Asymptotic Variance of $T(x)$

In order to construct meaningful distribution approximations using the asymptotic moments of this section, we now examine the variance of link travel time as the length of the link tends toward infinity. The limiting behavior of $Var[T(x)]$ is characterized using the K-dimensional generator matrix Q and the velocity matrix V. First we require some notation for the spectral representation of the matrix $V^{-1}Q$. Let $\eta_i, i = 1, 2, \ldots, K$, denote the K eigenvalues of $V^{-1}Q$, and let l_i (r_i), $i = 1, 2, \ldots, K$, denote their corresponding left (right) eigenvectors. Of the K eigenvalues, one eigenvalue is zero and the remaining $K-1$ are strictly negative. In particular, we note that the vector $\hat{p}$ is the left eigenvector corresponding to the zero eigenvalue. Using the remaining $K-1$ eigenvalues (and eigenvectors), the limiting behavior of $Var[T(x)]$ is characterized in Theorem 5.

Theorem 5

$$\lim_{x\to\infty} \frac{Var[T(x)]}{x} = -\frac{2}{pv} \sum_{i=2}^{K} \frac{1}{\eta_i} \frac{(pr_i)(l_i V^{-1} e)}{l_i r_i}. \tag{27}$$

Proof. By Equation (19), it can be shown that

$$m_1(x) = x/(pv) + O(x) \tag{28}$$

where $O(x)/x \to 0$ as $x \to \infty$. In like manner, it can be shown that $m_2(x)$ is of the form

$$\frac{m_2(x)}{x} = x/(pv)^2 + c(x) + \frac{O_1(x)}{x}, \tag{29}$$

where $c(x)$ is a function that depends on x and $O_1(x)/x \to 0$ as $x \to \infty$. Let

$$\psi(x) = \frac{Var[T(x)]}{x} = \frac{m_2(x)}{x} - \frac{(m_1(x))^2}{x}$$

and define its Laplace-Stieltjes transform (LST) as

$$\tilde{\psi}(s) = \int_0^\infty e^{-sx} d\psi(x). \tag{30}$$

For sufficiently large x, the function $\psi(x)$ can be written as

$$\psi(x) = c(x) - 2O(x)/(pv). \tag{31}$$

Taking the LST of both sides of Equation (31) gives

$$\tilde{\psi}(s) = \tilde{c}(s) - 2\tilde{O}(s)/(pv) \tag{32}$$

where $\tilde{\psi}(s)$ is defined as in Equation (30). Next, we compute the transforms $\tilde{c}(s)$ and $\tilde{O}(s)$. By taking the LST of both sides of Equation (28) and rearranging terms,

$$\tilde{O}(s) = \tilde{m}_1(s) - \frac{1}{(pv)s}$$

where $\tilde{m}_1(s)$ is the LST of $m_1(x)$ so that

$$\tilde{O}(s) = z_0(sV - Q)^{-1}e - \frac{1}{(pv)s} \tag{33}$$

and z_0 is the initial distribution of the environment process. Now to compute $\tilde{c}(s)$, we take the LST of both sides of Equation (29) and rearrange terms to obtain

$$\begin{aligned} \tilde{c}(s) &= s\tilde{m}_2(s) - \frac{2}{s(pv)^2} - LST\left\{\frac{O_1(x)}{x}\right\} \\ &= 2z_0 s(sV - Q)^{-2}e - \frac{2}{s(pv)^2} - LST\left\{\frac{O_1(x)}{x}\right\}, \end{aligned} \tag{34}$$

where $LST\{h(\cdot)\}$ denotes the LST of the function $h(\cdot)$. Substituting Equations (34) and (33) into Equation (32) and letting $s \to 0$ yields

$$\begin{aligned} \lim_{s\to 0}\tilde{\psi}(s) &= \lim_{s\to 0} 2z_0(sV - Q)^{-1}\left[s(sV - Q)^{-1} - I(pv)^{-1}\right]e - LST\left\{\frac{O_1(x)}{x}\right\} \\ &= \lim_{s\to 0}\frac{2}{pv}\left(p(sV - Q)^{-1}e - \frac{1}{spv}\right). \end{aligned}$$

Now let η_i denote one of the $K-1$ strictly negative eigenvalues of $V^{-1}Q$ and let l_i (r_i), $i = 2, \ldots, K$ denote the corresponding left (right) eigenvectors of $V^{-1}Q$. It can be shown (see Asmussen [3]) that, using the spectral representation of $V^{-1}Q$,

$$p(sV - Q)^{-1}e = \frac{1}{spv} + \sum_{i=2}^{K} \left(\frac{1}{s - \eta_i} \right) \frac{1}{l_i r_i} (pr_i)(l_i V^{-1} e)$$

where we omit the zero eigenvalue and its corresponding eigenvector. Hence, we obtain

$$\begin{aligned} \lim_{x \to \infty} \frac{Var[T(x)]}{x} &= \frac{2}{(pv)} \lim_{s \to 0} \left(p(sV - Q)^{-1}e - \frac{1}{spv} \right) \\ &= -\frac{2}{pv} \sum_{i=2}^{K} \frac{1}{\eta_i} \frac{(pr_i)(l_i V^{-1} e)}{l_i r_i} \end{aligned}$$

■

5 Numerical Examples

In this section, the performance of the analytical results of sections 3 and 4 are demonstrated on two numerical examples. The numerical transform inversions, obtained by using the algorithm of Abate and Whitt [1], are validated via Monte-Carlo simulation of link travel times under the assumption that vehicle speed is modulated by a continuous-time Markov chain.

Example 1: Ground Transportation Problem

Consider a vehicle traversing a roadway segment of length x miles. Due to time-variant traffic factors (e.g. flow and density), the speed of the vehicle may be categorized in one of 10 distinct ranges. Then the random environment process is a general, 10-state CTMC with state space $S = \{1, 2, \ldots, 10\}$. When the environment is in state i, the speed of the vehicle is $V_i = 75/i$ for $i \in S$. The off-diagonal entries of the generator matrix, $Q = [q_{ij}]_{i,j \in S}$, are distributed uniformly on the interval (200, 400) and the units are 1/hr. If the CTMC is currently in state $i \in S$, the process transitions to state $j \in S \setminus \{i\}$ with probability $q_{ij}/(-q_{ii})$. It is arbitrarily assumed that, with probability 1, the system starts in state 1 at

time 0. The off-diagonal entries of Q were computed by generating a uniform variate on the interval (0,1) and translating each entry so that it lies in the interval (200, 400), which is chosen arbitrarily. Table 2 displays the numerical results for this example.

Table 2: Lower moments for vehicle link travel time.

x (mi)	Measure	Transient	Simulated	Asymptotic
0.25	$m_1(x)$	0.647889	0.647530	0.681060
	$m_2(x)$	0.439642	0.439307	0.463843
0.50	$m_1(x)$	1.317198	1.318089	1.348891
	$m_2(x)$	1.779084	1.781252	1.819507
1.00	$m_1(x)$	2.658887	2.657945	2.691188
	$m_2(x)$	7.158717	7.153180	7.242492
5.00	$m_1(x)$	12.977443	12.978734	13.005931
	$m_2(x)$	168.807281	168.842391	169.154242
10.00	$m_1(x)$	26.218836	26.224240	26.249637
	$m_2(x)$	688.289778	688.571303	689.043454

Example 2: Variance Calculations

The purpose of this example is to demonstrate the variance calculations using Equation (27). We assume the random environment process is a general, 5-state CTMC with state space $S = \{1, 2, \ldots, 5\}$. When the environment is in state i, the speed of the vehicle is $V_i = 75/i$ for $i \in S$. The off-diagonal entries of the generator matrix, $Q = [q_{ij}]_{i,j \in S}$, are distributed uniformly on the interval (20, 60) with units 1/hr. If the CTMC is currently in state $i \in S$, the process transitions to state $j \in S \setminus \{i\}$ with probability $q_{ij}/(-q_{ii})$. It is arbitrarily assumed that, with probability 1, the system starts in state 1 at time 0. The off-diagonal entries of Q were computed by generating a uniform variate on the interval (0,1) and translating each entry so that it lies in the interval (20, 60). Table 3 displays numerical results comparing values obtained from Equation (27) and Monte-Carlo simulation.

Table 3: Asymptotic variance results.

x	Simulated	Asymptotic
1.00	0.232525	0.408426
5.00	2.111038	2.042130
10.00	3.968300	4.084261
50.00	20.491260	21.030564
100.00	41.294269	41.141287

6 Conclusions

We have presented a fluid queueing model for implicitly incorporating the time dependence of speed for a vehicle traversing a link of length x by considering a random environment process that evolves stochastically over time. When the environment process is known to be a continuous-time Markov chain, an explicit expression is obtained for the rth moment of the link travel time. The derived expression gives exact results when the transform can be algebraically inverted, and very accurate approximations with numerical Laplace transform inversion. Moreover, the transform expressions give rise to useful, asymptotic approximations in the form of limit theorems for the mean, second moment and variance of the random link travel time.

There are several real-world contexts that motivate the study of a vehicle traversing a random environment. The generality of our model allows for the exact analysis of the link travel time moments in a variety of transportation settings. In this work, we were primarily motivated by the need for computationally expedient measures that may be directly applied to stochastic transportation network problems. It is clear that the technique can be easily modified and extended to accommodate a number of different problem settings.

The techniques of this paper can potentially be used to construct three types of distributions for the random time to traverse a link of length x. First, it is always possible to solve for the matrix $H(x,t)$ via transform techniques and perform numerical inversion in two dimensions to obtain an approximate distribution. However, the numerical inversion process

is computationally intensive, particularly when K is large. An alternative is to construct surrogate transient or asymptotic normal approximations. Transient normal approximations should utilize the transient moment results of section 3 for the mean and variance of the distribution. Asymptotic normal approximations should use the limiting results for the mean and variance of link travel time (section 4). The appropriate choice of distribution will depend on the computation time or accuracy required by the algorithm in which the link travel time is used. The simpler, parametric distributions, particularly normal approximations, will be useful since look-up tables can be utilized to obtain cumulative distribution function values. The model can be suitably extended to more general environment processes (such as semi-Markov processes), depending upon the application.

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