Errata in Chapter 9

On page 555, part of Remark 23 the equation in the middle of the page:

$$\tilde{H}_{ij}(0,w) = \sum_{k \in S, k \neq i} \tilde{H}_{kj}(0,w) \left(\frac{q_{ik}}{-q_{ii}} \right) \frac{q_{ik}}{q_{ik}+w}$$

should be changed to

$$\tilde{H}_{ij}(0,w) = \sum_{k \in S, k \neq i} \tilde{H}_{kj}(0,w) \left(\frac{q_{ik}}{-q_{ii}} \right) \frac{-q_{ii}}{-q_{ii}+w}.$$

That directly affects the solution to Problem 92 on pages 563-566. However, it does *not* affect solution to problem 95 on page 572 where it is used.

Solution to problem 92
(changes marked in RED).

Solution. At $t = 0$ water level just crosses over from nominal to concerning. Using the same notation as in Problem **??** for $X(t)$ and $Z(t)$, we have $X(0) = 20$ and $Z(0) = 1$. Let T be the time when the water level crosses back to becoming nominal from concerning, i.e.

$$T = \inf\{t > 0 : X(t) = 20\}.$$

To compute $E[T]$, we follow the analysis in Remark **??** and use $H_{ij}(x,t) = P\{T \le t, Z(T) = j | X(0) = x, Z(0) = i\}$ in particular its LST w.r.t. t, $\tilde{H}_{ij}(x,w)$. To obtain $E[T]$, the expected number of days from $t = 0$ for the water level to return to nominal values, we use

$$E[T] = (-1)\frac{d}{dw} \sum_{j=1}^{5} \tilde{H}_{1j}(20,w)$$

at $w = 0$. To compute $\frac{d}{dw}\tilde{H}_{ij}(x,w)$ at $w = 0$ here too we consider a very small $h > 0$ and obtain it approximately as $\frac{\tilde{H}_{ij}(x,h) - \tilde{H}_{ij}(x,0)}{h}$. Now, to evaluate $\tilde{H}_{ij}(x,h)$ and $\tilde{H}_{ij}(x,0)$, we can write down from Equation (**??**), for $j = 1, 2, 3, 4, 5$,

$$\begin{bmatrix} \tilde{H}_{1j}(x,w) \\ \tilde{H}_{2j}(x,w) \\ \tilde{H}_{3j}(x,w) \\ \tilde{H}_{4j}(x,w) \\ \tilde{H}_{5j}(x,w) \end{bmatrix} = a_{1,j}(w)e^{S_1(w)x}\phi_1(w) + a_{2,j}(w)e^{S_2(w)x}\phi_2(w) + \ldots + a_{5,j}(w)e^{S_5(w)x}\phi_5(w) \qquad (1)$$

where $a_{i,j}(w)$, $S_j(w)$ and $\phi_j(w)$ values need to be determined for $w = 0$ and $w = h$ for some small h.

We can obtain $S_j(w)$ for $j = 1, 2, 3, 4, 5$ as the scalar solutions to the characteristic equation

$$det(DS(w) - wI + Q) = 0.$$

But this is identical to that in Problem **??**. Likewise $\phi_j(w)$ can be computed as the colimn vectors that satisfy

$$S_j(w)D\phi_j(w) = (wI - Q)\phi_j(w)$$

which is also identical to that in Problem **??**. Thus refer to Problem **??** for $\phi_j(w)$ and $S_j(w)$ for $j = 1, 2, 3, 4, 5$ at $w = 0$ and $w = h$. What remains in Equation (**??**) are the $a_{i,j}(w)$ values for $w = 0$ and $w = h$. For that refer back to the approach in Remark **??**. First of all

$$\begin{bmatrix} \tilde{H}_{1j}(x,w) \\ \tilde{H}_{2j}(x,w) \\ \tilde{H}_{3j}(x,w) \\ \tilde{H}_{4j}(x,w) \\ \tilde{H}_{5j}(x,w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $j = 1, 2$ since the first passage time can never end in states 1 or 2 as the drift is negative in those states (only when the drift is positive is is possible to cross over into a particular buffer content level from below). Thus we only need $a_{i,j}(w)$ values for $i = 1, 2, 3, 4, 5$ and $j = 3, 4, 5$.

Of those fifteen unknown $a_{i,j}(w)$ values, nine can be obtained through the following boundary conditions:

$$\tilde{H}_{33}(20,\ w) = 1, \tilde{H}_{34}(20,\ w) = 0, \tilde{H}_{35}(20,\ w) = 0,$$
$$\tilde{H}_{44}(20,\ w) = 1, \tilde{H}_{43}(20,\ w) = 0, \tilde{H}_{45}(20,\ w) = 0,$$
$$\tilde{H}_{55}(20,\ w) = 1, \tilde{H}_{53}(20,\ w) = 0 \quad \text{and} \quad \tilde{H}_{54}(20,\ w) = 0.$$

For the remaining six unknowns we use for $j = 3, 4, 5$

$$\begin{aligned} \tilde{H}_{1j}(0,w) &= \sum_{k=2}^{5} \tilde{H}_{kj}(0,w) \left(\frac{q_{1k}}{-q_{11}} \right) \frac{-q_{11}}{-q_{11}+w} \\ \tilde{H}_{2j}(0,w) &= \tilde{H}_{1j}(0,w) \left(\frac{q_{21}}{-q_{22}} \right) \frac{-q_{22}}{-q_{22}+w} + \sum_{k=3}^{5} \tilde{H}_{kj}(0,w) \left(\frac{q_{2k}}{-q_{22}} \right) \frac{-q_{22}}{-q_{22}+w} \end{aligned}$$

where q_{ij} corresponds to the element in the i^{th} row and j^{th} column of Q.

Solving the above fifteen equations we get for $w = 0$,

$$\begin{bmatrix} a_{1,3}(0) & a_{2,3}(0) & a_{3,3}(0) & a_{4,3}(0) & a_{5,3}(0) \\ a_{1,4}(0) & a_{2,4}(0) & a_{3,4}(0) & a_{4,4}(0) & a_{5,4}(0) \\ a_{1,5}(0) & a_{2,5}(0) & a_{3,5}(0) & a_{4,5}(0) & a_{5,5}(0) \end{bmatrix} =$$

$$\begin{bmatrix} 0.0093 \times 10^{-3} & -0.2211 \times 10^{-4} & -0.2109 & -0.0033 & -0.0014 \\ 0.1326 \times 10^{-3} & 0.1117 \times 10^{-4} & -0.8945 & -0.0466 & -0.0208 \\ -0.1419 \times 10^{-3} & 0.1094 \times 10^{-4} & -1.1307 & 0.0500 & 0.0223 \end{bmatrix}.$$

Also, for $w = h = 0.000001$, the values of $a_{i,j}(h)$ are the same as that when $w = 0$ to the first few significant digits. Hence we do not present that here.

Now using $a_{i,j}(w)$, $S_i(w)$ and $\phi_i(w)$ values for $i = 1, 2, 3, 4, 5$ and $j = 3, 4, 5$ at $w = 0$ and $w = h$ in Equation (1) we can compute $\tilde{H}_{ij}(x, w)$. In particular for $x = 20$ (which is what we need here) we get

$$\begin{bmatrix} \tilde{H}_{13}(20,0) & \tilde{H}_{14}(20,0) & \tilde{H}_{15}(20,0) \\ \tilde{H}_{23}(20,0) & \tilde{H}_{24}(20,0) & \tilde{H}_{25}(20,0) \\ \tilde{H}_{33}(20,0) & \tilde{H}_{34}(20,0) & \tilde{H}_{35}(20,0) \\ \tilde{H}_{43}(20,0) & \tilde{H}_{44}(20,0) & \tilde{H}_{45}(20,0) \\ \tilde{H}_{53}(20,0) & \tilde{H}_{54}(20,0) & \tilde{H}_{55}(20,0) \end{bmatrix} = \begin{bmatrix} 0.1583 & 0.3995 & 0.4422 \\ 0.1497 & 0.3913 & 0.4590 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The values of $\tilde{H}_{ij}(20, h)$ for some small h do not differ in the first four significant digits from the corresponding $\tilde{H}_{ij}(20, 0)$ values for $i = 1, 2, 3, 4, 5$ and $j = 3, 4, 5$, hence they are not reported.

Thus we have the expected number of days from $t = 0$ (with initial water level $X(0) = 20$ as well as initial environmental condition $Z(0) = 1$) for the water level to become nominal as

$$E[T] = (-1)\frac{d}{dw}\sum_{j=3}^{5}\tilde{H}_{1j}(20,w)|_{w=0} = -\lim_{h\to 0}\sum_{j=3}^{5}\frac{\tilde{H}_{1j}(20,h) - \tilde{H}_{1j}(20,0)}{h}$$

which is approximately 15.593 days by using $h = 0.000001$. ∎